

## Chapter 4

### Exercise 4.1

Methods A and B are genuine proof methods. Method C is not because there is no simple mechanical check of whether a sentence occurs in some logic textbook.

### Exercise 4.2

Method A is complete, but not sound. Everything that's K-valid is provable with the method, but so is everything that's not K-valid.

Method B is sound, but not complete. Since every instance of  $\Box(A \vee \neg A)$  is K-valid, everything that is provable with method B is K-valid. But many K-valid sentences (e.g.,  $p \rightarrow p$ ) aren't provable with method B.

Method C is neither sound nor complete. It is not sound because many K-invalid sentences figure in logic textbooks. It is not complete because there are infinitely many K-valid sentences almost all of which don't occur in any textbooks.

### Exercise 4.3

For  $A \rightarrow B$ : Suppose  $\beta$  contains a node of the form  $A \rightarrow B (\omega)$  and the branch is split into two, with  $\neg A (\omega)$  appended to one end and  $B (\omega)$  to the other. Since the expanded node is a correct statement about  $M$  under  $f$ , we have  $M, f(\omega) \models A \rightarrow B$ . By clause (e) of definition 3.2, it follows that either  $M, f(\omega) \not\models A$  or  $M, f(\omega) \models B$ . By clause (b), this means that either  $M, f(\omega) \models \neg A$  or  $M, f(\omega) \models B$ . So at least one of the resulting branches also correctly describes  $M$ .

For  $\neg\Diamond A$ : Suppose  $\beta$  contains nodes of the form  $\neg\Diamond A (\omega)$  and  $\omega Rv$ , and the branch is extended by adding  $\neg A (v)$ . Since  $\neg\Diamond A (\omega)$  and  $\omega Rv$  are correct statements about  $M$  under  $f$ , we have  $M, f(\omega) \models \neg\Diamond A$  and  $f(\omega)Rf(v)$ . By clause (b) of definition 3.2,  $M, f(\omega) \models \neg\Diamond A$  implies  $M, f(\omega) \not\models \Diamond A$ . By clause (h), it follows that  $M, f(v) \models \neg A$ . So the extended branch correctly describes  $M$ .

### Exercise 4.4

Yes. The function  $f$  can map both 'w' and 'v' to  $w$ .

### Exercise 4.5

A sentence is K4-valid iff it is true at all worlds in all transitive Kripke models. We only need to check that the Transitivity rule is sound, in the sense that if a branch correctly describes a transitive model  $M$ , and the branch is extended by the Transitivity rule, then the resulting branch also correctly describes  $M$ . (The Transitivity rule allows adding a node  $\omega Rv$  to a branch that already contains nodes  $\omega Rv$  and  $vRv$ . If these nodes correctly describe a transitive model then so does  $\omega Rv$ .)

#### Exercise 4.6

For  $B \rightarrow C$ : If  $A$  is a conditional  $B \rightarrow C$ , then  $\beta$  contains either  $\neg B$  ( $\omega$ ) or  $C$  ( $\omega$ ). By induction hypothesis,  $M, \omega \models \neg B$  or  $M, \omega \models \neg C$ . Either way, clauses (b) and (e) of definition 3.2 imply that  $M, \omega \models A$ .

For  $\neg\Diamond B$ : If  $A$  is a negated diamond sentence  $\neg\Diamond B$ , then  $\beta$  contains a node  $\neg B$  ( $v$ ) for each world variable  $v$  for which  $\omega Rv$  is on  $\beta$  (because the tree is fully developed). By induction hypothesis,  $M, v \models \neg B$ , for each such  $v$ . By definition 4.2, it follows that  $M, v \models \neg B$  for all worlds  $v$  such that  $\omega Rv$ . By clauses (b) and (g) of definition 3.2, it follows that  $M, \omega \models A$ .

#### Exercise 4.7

We need to check that the model induced by an open branch on a fully developed K4-tree is transitive. (Suppose the model contains worlds  $w, v, u$  for which  $wRv$  and  $vRu$ . Then the Transitivity rule has been applied to the corresponding nodes on the branch, generating a node  $wRu$ . By definition 4.2,  $wRu$  holds in the induced model.)

#### Exercise 4.8

Suppose  $A$  is true at some world in some Kripke model. Then  $\neg A$  is K-invalid. Take any regular K-tree for  $\neg A$ . By observation 4.1, that tree is fully developed. By the soundness theorem for K-trees, the tree has an open branch. Let  $M$  be the model induced by some such branch  $\beta$ . Then  $M$  is acyclical. This is because the only rules that allow adding a node  $\omega Rv$  to a branch of a K-tree are the rules for expanding  $\Diamond A$  and  $\neg\Box A$  nodes. In both cases, the rule requires that the relevant world variable  $v$  is new on the branch. (Call this the *novelty requirement*. Now suppose the accessibility relation in  $M$  has a cycle  $\omega_1 R\omega_2, \omega_2 R\omega_3, \dots, \omega_{n-1} R\omega_n, \omega_n R\omega_1$ . Each of these facts about  $R$  must correspond to a node on  $\beta$ . Of these nodes, the one that was

added last (to  $\beta$ ) violates the novelty requirement. So  $M$  is acyclical.

By the Completeness Lemma, the target sentence  $\neg\neg A$  is true at world  $w$  in  $M$ . So  $A$  is true at  $w$  in  $M$ . So  $A$  is true at some world in some acyclical model.

#### Exercise 4.9

The S5 rules are not sound with respect to K-validity. For example,  $\Box p \rightarrow p$  is provable with the S5 rules, but it isn't K-valid. The rules are, however, complete with respect to K-validity. This follows from the completeness of the S5 rules and the fact that every K-valid sentence is S5-valid (observation 3.1).

#### Exercise 4.10

We need to show that everything that's derivable in the axiomatic calculus for S4 is true at every world in every transitive and reflexive Kripke model. From the soundness proof for K, we know that all instances of (Dual) and (K) are true at every world in every Kripke model. From observation 3.2, we know that all instances of (T) are true at every world in every reflexive Kripke model. From observation 3.3, we know that all instances of (4) are true at every world in every transitive Kripke model. So all axioms in the S4-calculus are valid in the class of transitive and reflexive Kripke frames. Since (CPL) and (Nec) preserve validity in any class of Kripke frames, it follows that everything that's derivable in the S4-calculus is valid in the class of transitive and reflexive frames.

#### Exercise 4.11

(a), (b), and (c) are K-consistent, (d) is not.

#### Exercise 4.12

We have to show that all S5-valid sentences are provable in the axiomatic calculus for S5, which extends the calculus for T by the axiom schemas  $\Box A \rightarrow \Box\Box A$  and  $\Diamond A \rightarrow \Box\Diamond A$ . (The second schema alone would be sufficient, as I mentioned in chapter 1, but it doesn't hurt to have the first.) The argument is by contraposition: We suppose that some sentence is not S5-provable and show that it is not S5-valid.

Suppose  $A$  is not S5-provable. Then  $\{\neg A\}$  is S5-consistent. It follows by Lindenbaum's Lemma that  $\{\neg A\}$  is included in some maximal S5-consistent set  $\Gamma$ . By

definition of canonical models, this set is a world in the canonical model  $M_{S5}$  for S5. Since  $\neg A$  is in  $\Gamma$ , it follows from the Canonical Model Lemma that  $M_{S5}, \Gamma \models \neg A$ . So  $M_{S5}, S \not\models A$ .

It remains to show that the accessibility relation in  $M_{S5}$  is reflexive, transitive, and symmetric (for every such relation is an equivalence relation, and a sentence is S5-validity iff it is valid in the class of Kripke models whose accessibility relation is an equivalence relation).

By definition, a world  $v$  in a canonical model is accessible from  $w$  iff whenever  $\Box A \in w$  then  $A \in v$ . Since the worlds in  $M_{S5}$  are maximal S5-consistent sets of sentences, and every such set contains every instance of the (T)-schema  $\Box A \rightarrow A$ , there is no world in  $M_{S5}$  that contains  $\Box A$  but not  $A$ . So every world in  $M_{S5}$  has access to itself.

For transitivity, suppose for some worlds  $w, v, u$  in  $M_{S5}$  we have  $wRv$  and  $vRu$ . We need to show that  $wRu$ . Given how  $R$  is defined in  $M_{S5}$ , we have to show that  $u$  contains all sentences  $A$  for which  $w$  contains  $\Box A$ . So let  $A$  be an arbitrary sentence for which  $w$  contains  $\Box A$ . Since every world in  $M_{S5}$  contains every instance of  $\Box A \rightarrow \Box \Box A$ , we know that  $w$  also contains  $\Box \Box A$ . From  $wRv$ , we can infer that  $v$  contains  $\Box A$ . And from  $vRu$ , we can infer that  $u$  contains  $A$ .

For symmetry, suppose for some worlds  $w, v$  in  $M_{S5}$  we have  $wRv$  and not  $vRw$ . Given how  $R$  is defined, this means that there is some sentence  $A$  for which  $\Box A$  is in  $v$  but  $\neg A$  is in  $w$ . Since  $w$  contains the T-provable sentence  $\neg A \rightarrow \Diamond \neg A$  and the (5)-instance  $\Diamond \neg A \rightarrow \Box \Diamond \neg A$ , it also contains  $\Box \Diamond \neg A$ . So  $v$  contains  $\Diamond \neg A$ . This contradicts the assumption that  $v$  is S5-consistent, given that  $v$  contains  $\Box A$ .

#### Exercise 4.13

(a) Method A from exercise 4.1 is sound and complete for  $X$ . (b) No set of  $\mathcal{L}_M$ -sentences is  $X$ -consistent, but every Kripke model must have at least one world.

#### Exercise 4.14

Let  $\Gamma$  is an infinite set of  $\mathcal{L}_M$ -sentences. If  $\Gamma$  is K-satisfiable then obviously every finite subset of  $\Gamma$  is satisfiable as well. For the converse direction, assume  $\Gamma$  is not K-satisfiable: There is no world in any Kripke model at which all members of  $\Gamma$  are true. Then there is no world in any Kripke model at which all members of  $\Gamma$  are true while  $p \wedge \neg p$  is false. So  $\Gamma \models p \wedge \neg p$ . By the compactness theorem, it follows that

there is a finite subset  $\Gamma^-$  for which  $\Gamma^- \models p \wedge \neg p$ . If  $\Gamma^- \models p \wedge \neg p$  then there is no world in any Kripke model at which all members of  $\Gamma^-$  are true while  $p \wedge \neg p$  is false. Since  $p \wedge \neg p$  is false at every world in every Kripke model, it follows that there is no world in any Kripke model at which all members of  $\Gamma^-$  are true. This shows that if  $\Gamma$  is not K-satisfiable then there is a finite subset ( $\Gamma^-$ ) of  $\Gamma$  that is not K-satisfiable. Conversely, if every finite subset of  $\Gamma$  is K-satisfiable then  $\Gamma$  is K-satisfiable.

**Exercise 4.15**

Suppose there is a proof of  $\neg \Box(p \wedge \neg p)$ . By (CPL), we can infer  $\Box(p \wedge \neg p) \rightarrow (p \wedge \neg p)$ , because  $A \rightarrow B$  is a truth-functional consequence of  $\neg A$ . By (Nec), we get  $\Box(\Box(p \wedge \neg p) \rightarrow (p \wedge \neg p))$ . By (GL) and *modus ponens* (an instance of (CPL)), we can derive  $\Box(p \wedge \neg p)$ .