1 Modal Operators

1.1 Boxes and diamonds

Modal logic is an extension of propositional and predicate logic that is widely used (across many disciplines) to reason about possibility and necessity, obligation and permission, the flow of time, the processing of computer programs, and a range of other topics. Each of these applications begins by adding new symbols to the formal language of classical propositional or predicate logic. Before we explore such additions, let's briefly review why we use formal languages in the first place.

When reasoning about a given topic, we want to make sure that the stated conclusions really follow from the stated premises. If they do, we say that the reasoning is valid. A little more precisely, an argument is **valid** if there is no conceivable situation in which the premises are true while the conclusion is false.

Here is an example of a valid argument.

> All myriapods are oviparous.
> Some arthropods are myriapods.
> Therefore: some arthropods are oviparous.

You can tell that this argument is valid even if you don't understand the zoological terms, because every argument of the same **logical form** is valid. The logical form of the above argument might be expressed as follows.

> All F are G.
> Some H are F.
> Therefore: some H are G.

No matter what descriptive terms you plug in for F, G, and H, you get a valid argument. The argument about myriapods is therefore not just valid, but **logically valid** – valid in virtue of its form.

In natural languages like English, the logical form of sentences is not always transparent. 'All dogs barked at a tree' can mean either that there is a single tree at which all dogs barked, or that each dog barked at some tree or other. The two readings have different logical consequences, so it would be good to keep them apart. Also, the meaning of logical expressions ('all', 'some', 'and', etc.) in natural language is often unclear and complicated. 'Paul and Paula got married and had children' suggests that the marriage came before the children. In 'Paul went to the zoo and Paula stayed at home', the word 'and' does not seem to have this temporal meaning.

To get around these problems, we invent formal languages whose logical expressions have precise meanings and in which there are no ambiguities of logical form. If we want to evaluate natural-language arguments for logical validity, we first have to translate them into the formal language. (Sometimes an argument will be valid on one translation and invalid on another.) We can also reason directly in the formal language – perhaps to show that under a certain interpretation of the logical terms, a certain hypothesis logically follows from certain assumptions.

Returning to modal logic, consider the following argument.

> It might be raining.
> There is no doubt that we will get wet if it is raining.
> Therefore: we might get wet.

The argument looks valid. Indeed, any argument of this form is plausibly valid:

> It might be that A.
> There is no doubt that B if A.
> Therefore: it might be that B.

But it's hard to bring out the validity of these arguments in classical propositional or predicate logic. We need formal expressions corresponding to 'it might be that' and 'there is no doubt that'. The language of classical propositional or predicate logic does not have such expressions.

So let's add them. That is, let's invent a new formal language with two new logical symbols. It doesn't matter what these look like; a popular choice in modal logic is to use a diamond $\Diamond$ and a box $\Box$. (We will use various other symbols in later chapters.) Let's assume that the diamond formalizes 'it might be that', and the box 'there is no doubt that' or (equivalently) 'it is certain that'. We can then formalize the above argument as follows.

$$\begin{array}{l} \Diamond r \\ \Box(r \to w) \\ \hline \Diamond w \end{array}$$

Of course, merely adding new symbols doesn't help much. We also need to lay down some new rules for how to reason with these symbols. The rules should be motivated by what the symbols are supposed to mean. So we shall also assign a more precise meaning to the diamond and the box – just as classical logic assigns a precise meaning to the symbol $\wedge$ which may or may not exactly match the meaning of 'and' in English. We can then confirm that no matter what r and w mean, if $\Diamond r$ and $\Box(r \to w)$ are both true, then so is $\Diamond w$.

We have entered the realm of modal logic.

Historically, modal logic grew out of the study of necessity and contingency, which medieval logicians regarded as "modes of truth"; hence the name 'modal logic'. Today, the study of necessity and contingency is but one of many subfields within modal logic. To a first approximation, any part of logic that involves *non-truth-functional sentence operators* is part of modal logic.

By a **sentence operator** I mean an expression that combines with one or more sentences to create a new sentence. Classical propositional logic has the sentence operators $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\to$ (if-then), and $\leftrightarrow$ (if-and-only-if). In modal logic, we have further operators such as $\Diamond$ and $\Box$. (Sentence operators are also called 'connectives'.)

The sentence operators of classical propositional logic are all truth-functional. Remember that an operator is **truth-functional** if the truth-value of a complex sentence built with the use the operator is determined by the truth-value of its parts. For example, a conjunction $A \wedge B$ is true whenever A and B are both true, and false otherwise. If you know the truth-value of the conjuncts, you know the truth-value of the conjunction.

Things are different for our operators $\Diamond$ and $\Box$. The truth-value of $\Diamond A$ and $\Box A$ is *not* determined by the truth-value of A. That is, there are conceivable scenarios in which two sentences A and B have the same truth-value whereas $\Diamond A$ and $\Diamond B$ (or $\Box A$ and $\Box B$) have different truth-values.

For example, let r translate ‘it is currently raining in Sydney’, and let t translate ‘2+2=5’. Reading the diamond as ‘it might be the case that’, it is easy to imagine a scenario in which $\Diamond r$ and $\Diamond\neg r$ are both true, while $\Diamond t$ is false. Since r and $\neg r$ have opposite truth values, one of them must be false. If r is false, then r and t have the same truth-value while $\Diamond r$ and $\Diamond t$ have different truth-values; if $\neg r$ is false, then $\neg r$ and t have the same truth-value while $\Diamond\neg r$ and $\Diamond t$ have different truth-values. Either way, we have a counterexample to the truth-functionality of the diamond.

Exercise 1.1

Which of these English expressions are truth-functional?

(a) It used to be the case that ...
(b) It is widely known that ...
(c) It is false that ...
(d) It is necessary that ...
(e) I can see that ...
(f) God believes that ...
(g) Either 2+2=4 or it is practically feasible that ...

The meaning of truth-functional operators can be specified by a truth-table. All you need to know to understand $\land$ is that $A \land B$ is true whenever A and B are both true (and false otherwise). The meaning of $\Diamond$ and $\Box$, by contrast, cannot be given by a truth-table. The standard approach to define the meaning of modal operators instead involves the concept of possible worlds. Roughly, we’ll interpret $\Diamond A$ as saying that A is true at some possible world, and $\Box A$ as saying that A is true at all possible worlds.

Much more on this later.

1.2 Reasoning with boxes and diamonds

Let’s be clear about our formal language. If we add the box and the diamond to the language of classical propositional logic, we get the **standard language of modal propositional logic**, for short, $\mathfrak{L}_M$. We will stick with propositional logics until chapter 9, when we turn to modal predicate logic.

The sentences of $\mathfrak{L}_M$ are defined as follows.

1. Every sentence letter $p, q, r, \ldots$ is an $\mathfrak{L}_M$-sentence.
2. If A is an $\mathfrak{L}_M$-sentence, then so are $\neg A$, $\Diamond A$, and $\Box A$.
3. If A and B are $\mathfrak{L}_M$-sentences, then so are $(A \land B)$, $(A \lor B)$, $(A \to B)$ and $(A \leftrightarrow B)$.
4. Nothing else is an $\mathfrak{L}_M$-sentence.

As usual, outermost parentheses are omitted when displaying sentences. So $p \land q$ is treated as an abbreviation of $(p \land q)$.

Exercise 1.2

Which of these are $\mathfrak{L}_M$-sentences?

(a) p
(b) $\Diamond$
(c) $\Diamond p \lor (\Box p \to p)$
(d) $\Box\Box p$
(e) $\Box A \to A$
(f) $(\Diamond r \land \Diamond qr) \land \Diamond\Box\Diamond\Box p$

The sentence letters of $\mathfrak{L}_M$ don't have a determinate meaning. It doesn't make sense to ask, without further information, whether p or $\Diamond q$ are true sentences of $\mathfrak{L}_M$. However, it does make sense to ask whether a given $\mathfrak{L}_M$-sentence logically follows from others, for this only depends on the logical form of the relevant sentences.

For example, in the previous section I claimed that if the diamond formalizes 'it might be that' and the box 'there is no doubt that', and if r and w translate 'it is raining' and 'we will get wet' respectively, then $\Diamond r$ and $\Box(r \to w)$ logically entail $\Diamond w$. But if the entailment is genuinely a matter of logic, then it doesn't depend on the meaning of r and w. No matter what meaning we give to r and w, $\Diamond r$ and $\Box(r \to w)$ will entail $\Diamond w$.

Logicians often use the symbol '$\models$' (the "double-barred turnstile") to express logical consequence. The claim that $\Diamond r$ and $\Box(r \to w)$ logically entail $\Diamond w$ can then be expressed as follows:

$$\Diamond r, \Box(r \to w) \models \Diamond w.$$

Intuitively, this says that there is no conceivable scenario in which $\Diamond r$ and $\Box(r \to w)$ are true while $\Diamond w$ is false, no matter what meaning is given to r and w.

Note that '$\models$' is not a symbol of $\mathfrak{L}_M$, nor does $\mathfrak{L}_M$ have a comma. The comma and the double-barred turnstile belong to the **meta-language** we use to talk about the **object language** $\mathfrak{L}_M$. (The rest of our meta-language is mostly English.) The turnstile is used to express a certain relationship between $\mathfrak{L}_M$-sentence; it is not part of any $\mathfrak{L}_M$-sentence.

Clearly, if $\Diamond r$ and $\Box(r \to w)$ logically entail $\Diamond w$, then this doesn't depend on the choice of the letters r and w. We also have, for example:

$$\Diamond p, \Box(p \to q) \models \Diamond q$$

In general, it is part of the meaning of the turnstile (of 'logical consequence' or 'logical entailment') that uniformly replacing sentence letters by other sentence letters does not change facts about logical entailment.

We can generalize even further. Arguably, if the inference from $\Diamond p$ and $\Box(p \to q)$ to $\Diamond q$ is logically valid, then it remains valid if we replace the sentence letters p and q by arbitrary sentences, not just by other sentence letters. For example, if it might be cold and windy, and there is no doubt that the picnic does not take place if it is cold and windy, we can infer that it might be that the picnic does not take place:

$$\Diamond(c \wedge w), \Box((c \wedge w) \to \neg p) \models \Diamond \neg p$$

We can summarize the general pattern by a **schema**:

$$\Diamond A, \Box(A \to B) \models \Diamond B$$

Here 'A' and 'B' are placeholders for arbitrary $\mathfrak{L}_M$-sentences. The schematic statement means that if you plug in any $\mathfrak{L}_M$-sentences for A and B, the sentences to the left of the turnstile logically entail the sentence on the right. Anything that results from a schema by uniformly replacing the placeholders 'A', 'B', 'C', etc. with object-language sentences is called an **instance** of the schema. "Uniformly" means that the same schematic letter ('A', 'B', 'C', etc.) is always replaced by the same object-language sentence. (It is not required that different schematic letters are replaced by different object-language sentence.)

Exercise 1.3

Which of the following (meta-language) statements are instances of $\Diamond A, \Box(A \to B) \models \Diamond B$?

(a) $\Diamond p, \Box(p \to (q \wedge r)) \models \Diamond(q \wedge r)$
(b) $\Diamond p, \Box(p \to p) \models \Diamond p$
(c) $\Diamond(A \wedge B), \Box((A \wedge B) \to C) \models \Diamond C$
(d) $\Diamond\Box r, \Box(\Diamond\Box r \to \neg(p \wedge q)) \models \Diamond\neg(p \wedge q)$
(e) $\Diamond\Diamond p, \Box(\Diamond p \to \neg\Box(q \wedge r)) \models \Diamond\neg\Box(q \wedge r)$

So far, we have met one inference pattern that appears to be valid on the interpretation we have given to the box and the diamond: from $\Diamond A$ and $\Box(A \to B)$ one can infer $\Diamond B$. Let's look at other such patterns.

We can plausibly assume that modal propositional logic inherits the valid inference patterns from non-modal propositional logic. For example, any instance of $A \wedge B$ plausibly entails the relevant instance of A, even if the instances contain modal operators. This makes our modal logic an **extension** of classical propositional logic.

A useful fact about entailment in classical logic, which carries over to extensions of classical logic, is this:

Observation 1.1: If Γ ('gamma') is a list of sentences and A and B are sentences, then

$$\Gamma, A \models B \text{ iff } \Gamma \models A \to B$$

Proof. For a rigorous proof of observation 1.1, we would need a precise definition of the turnstile. (I will give such a definition in chapter 2.) But the intuitive reason is easy enough to understand. Let me explain the left-to-right direction, that if $\Gamma, A \models B$ then $\Gamma \models A \to B$. The right-to-left direction is similar.

The argument is by contraposition. I will show that if $\Gamma \models A \to B$ is *not* the case, then $\Gamma, A \models B$ is not the case either. So assume that for some sentences A, B, and some list Γ, it is not the case that $\Gamma \models A \to B$. This means that there is a conceivable scenario in which the sentences in Γ are all true while $A \to B$ is false, on some interpretation of the sentence letters. By the truth table for the material conditional, $A \to B$ is false only if A is true and B is false. So in the relevant scenario, the sentences in Γ and A are true and B is false (under the given interpretation of the sentence letters). And then $\Gamma, A \models B$ is false, for $\Gamma, A \models B$ states that there is no conceivable scenario in which the sentences in Γ and A are all true while B is false, under any interpretation of the sentence letters. □

Observation 1.1 tells us that we can always move the turnstile to the left and put an arrow in its original position. For example, instead of

$$\Box(p \to q), \Diamond p \models \Diamond q$$

we can equivalently say

$$\Box(p \to q) \models \Diamond p \to \Diamond q.$$

We can even go one step further to

$$\models \Box(p \to q) \to (\Diamond p \to \Diamond q).$$

This says that $\Box(p \to q) \to (\Diamond p \to \Diamond q)$ logically follows from no premises at all: the sentence is true in all conceivable scenarios under all interpretations of the sentence letters. Sentences like this are called **logically true** or **(logically) valid**.

(So an *argument* is called valid if the conclusion follows from the premises, while a *sentence* is called valid if it follows from no premises.)

Make sure you don't confuse the arrow with the turnstile. For one, the two symbols belong to different languages. The arrow is part of the object-language $\mathfrak{L}_M$, while the turnstile is part of our meta-language. Moreover, the two symbols have very different meanings. In $\mathfrak{L}_M$, $p \to q$ is true iff either p is false or q is true. By contrast, $p \models q$ is true iff there is no conceivable scenario in which p is true and q is false, on any interpretation of the two letters. Nonetheless, there is an important connection between the arrow and the turnstile: $A \models B$ is *true* iff $A \to B$ is *valid*. This connection is generalised by observation 1.1.

A practical upshot of observation 1.1 is that instead of asking which sentences in $\mathfrak{L}_M$ entail which other sentences, we can equivalently ask which sentences are valid. This is how the question is often framed in modal logic. For example, our earlier claim that $\Box(A \to B), \Diamond A \models \Diamond B$ can be re-stated as the claim that all instances of the following schema are valid:

$$\Box(A \to B) \to (\Diamond A \to \Diamond B) \tag{$\mathbf{K}^*$}$$

If all instances of a schema are valid, we say that the schema itself is valid. (Note that $\mathbf{K}^*$ no longer contains a turnstile; the instances of the schema are simply $\mathfrak{L}_M$-sentences.)

1.3 Some modal schemas

The schema $\mathbf{K}^*$ is closely related to a more famous schema known as **K** (after Saul Kripke):

$$\Box(A \to B) \to (\Box A \to \Box B) \qquad (\mathbf{K})$$

On the present reading of the box as 'it is certain that', treating **K** as valid is to assume that if $A \to B$ and A are both certain, then it logically follows that B is also certain.

I will explain how **K** and $\mathbf{K}^*$ are related in a moment. First, I want to introduce two other schemas, connecting the box and the diamond.

$$\neg\Box A \leftrightarrow \Diamond\neg A \qquad (\mathbf{Dual1})$$

$$\neg\Diamond A \leftrightarrow \Box\neg A \qquad (\mathbf{Dual2})$$

Both of these are plausibly valid on our interpretation of the box and the diamond. Take **Dual1**. From left to right, this says that if it is not certain that A is true, then A might be false. From right to left, it states that if A might be false, then it is not certain that A is true. Similarly, from left to right, **Dual2** states that if it is not the case that A might be true, then it is certain that A is false; from right to left, it states that if it is certain that A is false, then it is not the case that A might be true. I hope you agree that these claims sound plausible.

Dual1 and **Dual2** are equivalent (in classical propositional logic) to the following schemas, as you should check:

$$\Box A \leftrightarrow \neg\Diamond\neg A \qquad (\mathbf{Dual1})$$

$$\Diamond A \leftrightarrow \neg\Box\neg A \qquad (\mathbf{Dual2})$$

So, if we wanted, we could define the box in terms of the diamond, or the diamond in terms of the box.

If two operators stand in the relationship expressed by **Dual1** and **Dual2**, they are called **duals** of each other. 'It might be that' and 'it is certain that' are plausibly duals, but so are many other pairs of expressions in natural language. In modal logic, there is a convention to use the symbols $\Box$ and $\Diamond$ only for concepts that are duals of each other.

Exercise 1.4

Find all pairs of duals among the following English expressions.

(a) It is necessary that ...
(b) It is impossible that ...
(c) It is possible that ...
(d) It is possibly not the case that ...
(e) It was the case that ...
(f) It will be the case that ...
(g) It has always been the case that ...
(h) It will always be the case that ...
(i) The law requires that ...
(j) The law does not require that ...
(k) The law allows that ...
(l) It is true that ...
(m) It is false that ...

Now I can explain how $\mathbf{K}^*$ is related to $\mathbf{K}$: the two schemas are plausibly equivalent, in the sense that the validity of one entails the validity of the other. Again, I'll only show one direction, because the other direction is similar. I'll show that if $\mathbf{K}$ is valid, then so is $\mathbf{K}^*$.

So suppose that $\mathbf{K}$ is valid; that is, every instance of $\mathbf{K}$ is a logic truth. Then so is every instance of

$$\Box(\neg A \to \neg B) \to (\Box\neg A \to \Box\neg B)$$

because every instance of this schema *is* an instance of $\mathbf{K}$.

To proceed, I need a further assumption. I'll assume that if two sentences are logically equivalent, then replacing one by the other in a more complex sentence does not affect whether the larger sentence is logically valid. On that assumption, we can replace $\neg A \to \neg B$ in the previous schema by $B \to A$:

$$\Box(B \to A) \to (\Box\neg A \to \Box\neg B)$$

Moreover, assuming that **Dual2** is valid, $\Box\neg A \to \Box\neg B$ is logically equivalent to $\neg\Diamond A \to \neg\Diamond B$, which in turn is equivalent to $\Diamond B \to \Diamond A$. So we get

$$\Box(B \to A) \to (\Diamond B \to \Diamond A),$$

which is obviously equivalent to $\mathbf{K}^*$.

Exercise 1.5

Spell out the converse argument, that if $\mathbf{K}^*$ is valid, then so is **K**.

Here are some other famous principles from modal logic; these are not entailed by **K**.

$$\Box A \to A \tag{\textbf{T}}$$

$$\Box A \to \Diamond A \tag{\textbf{D}}$$

$$\Box A \to \Box\Box A \tag{\textbf{4}}$$

$$\Diamond A \to \Box\Diamond A \tag{\textbf{5}}$$

A few comments on the names. The first schema is called **T** because its validity means that $\Box A$ implies the *truth* of A. The second is called **D** because it plays an important role in a branch of modal logic called *deontic logic*, where the box is read as 'it is obligatory that' and the diamond as 'it is permitted that'. The labels **4** and **5** allude to a list of logical "systems" discussed by C.I. Lewis in the late 1920s and early 1930s. **4** is the characteristic principle of the fourth system ('S4') in Lewis's list; **5** is the characteristic principle of the fifth system ('S5').

Which of these schemas should we accept as valid if we read the box as 'it is certain that' and the diamond as 'it might be that'?

Consider **T**. If it is certain that A, does it logically follow that A is true? Maybe not. What do you think?

The validity of **D** would mean that whenever it is certain that A, then A might be the case. Now, if it is certain that A, it would be odd to say that A *might* be the case.

But that something sounds odd doesn't mean that it is false. If we are certain that A, and the question arises whether A might be true, the correct answer is arguably 'yes', not 'no'. So I'd say that **D** is valid.

What about **4** and **5**? If something is certain, can we infer that it is certain that it is certain? If something might be the case, can we infer that it is certain that it might be the case? Again, the answers aren't obvious.

It gets worse. Consider the following schema, named after Peter Geach.

$$\Diamond\Box A \to \Box\Diamond A \tag{G}$$

If it might be the case that something is certain, is it certain that it might be the case? What does that even mean?

The kind of problem we here encounter comes up often in modal logic. We start with an intuitive concept, such as the concept that something might be the case. We represent this concept by a new symbol in a formal language. When we then consider which inferences involving the new symbol should count as valid, we realize that the intuitive concept with which we began does not give a clear answer. We need to sharpen the concept, giving it a clearer meaning, if we want to define its logic.

Exercise 1.6

Explain why, if **T** is valid, then so is the converse of **4**, $\Box\Box A \to \Box A$.

Exercise 1.7

Show that **T** is valid iff $A \to \Diamond A$ is valid, assuming the validity of **Dual1** and **Dual2**.

1.4 Flavours of modality

We have looked at one application of modal logic, in which we introduced sentence operators for 'it might be' and 'it is certain'. There are many other applications.

In philosophy and linguistics, a statement is classified as *modal* if it is about what *must* or *may* or *might* or *can* or *could have* been the case, in some sense of 'must', 'may', 'might', 'can', or 'could have'. So the following statements would be classified as modal.

(1) It must be raining.

(2) We might get wet.

(3) There can't be any water left.

(4) It can take years to earn someone's trust.

(5) It could have been raining.

(6) You can't go from Auckland to Sydney by train.

(7) In chess, you must be ruthless.

(8) You may leave now.

(9) You can take the 41 bus to get to the railway station.

Modal statements don't have to involve an auxiliary verb like 'might', 'must', or 'may'. We can, for example, use adjectives like 'possible' and 'necessary' to talk about what might or must be the case. Or we can use adverbs like 'possibly' and 'necessarily'. Verbs like 'have to' and 'ought to' also have a modal meaning, as do suffixes like '-ble' in 'comprehensible', 'legible', or 'edible'.

So the class of modal statements is grammatically diverse. In terms of meaning, we can distinguish at least three groups, often called *flavours* of modality.

The first group are statements about what is known, or entailed by the available evidence. Examples (1)–(3) fall into this category, at least on their most natural usage. This flavour of modality is called *epistemic* (from Greek *episteme*: 'knowledge').

A second flavour of modality, illustrated by examples (7)–(9), is concerned with rules, prescriptions, norms, or with what is required to achieve a certain goal. This flavour of modality is called *deontic* (from Greek *deontos*: 'of that which is binding').

Examples (4)–(6) would normally be understood as neither epistemic nor deontic. When I say that you can't go from Auckland to Sydney by train, I don't just mean that my information implies that you *don't* go from Auckland to Sydney by train; nor do I mean that you're not permitted to go, by some relevant norms. Rather, I mean that relevant circumstances in the world – such as the presence of an ocean between Auckland and Sydney – make it impossible for you to travel the journey by train. This flavour of modality is sometimes called *circumstantial*. It comes in many sub-flavours, depending on what kinds of circumstances are considered.

Each of these flavours of modality corresponds to a branch of modal logic.

Epistemic logic is a branch of modal logic that formalizes reasoning about knowledge and information. When we understood the diamond in $\mathfrak{L}_M$ as 'it might be that' and the box as 'it is certain that', we were doing epistemic logic. We will return to epistemic logic in chapter 5.

Deontic logic is another branch of modal logic, concerned with norms, permissions, and obligations. Standard deontic logic has a box-like sentence operator for 'it is obligatory that' (or 'it must be that', in the deontic sense), and a diamond-like operator for 'it is permissible that' (or 'it may be that', in the deontic sense). We will look at deontic logic in chapter 6.

A third branch of modal logic might be called *circumstantial logic*, but nobody uses that label. Some authors speak of *alethic modal logic* (from *aletheia*: 'truth'), but that label is also not used widely, and it is used for different things by different authors.

Two sub-flavours of circumstantial modality deserve special mention, because much philosophical work in modal logic concentrates on these sub-flavours.

The first sub-flavour is known as *metaphysical* modality. Metaphysical modality is concerned with what is compatible or incompatible with the nature of things. For example, many philosophers have the intuition that part of what it is to be water – part of the very nature of water – is to contain hydrogen. In philosophy jargon, this means that it is "metaphysically necessary" that water contains hydrogen. To formalize reasoning about metaphysical modality, we can introduce a box-like operator for metaphysical necessity, and a diamond-like operator for the dual concept of metaphysical possibility. The term 'alethic modal logic' is sometimes reserved for this sub-branch of circumstantial modal logic.

The other sub-flavour is variously known as *logical* or *absolute* or *unrestricted* modality. Here, the guiding intuition is that when we consider whether something is circumstantially possible – say, whether one could travel from Auckland to Sydney by train – we normally ignore various possibilities that are incompatible with conversationally relevant facts or circumstances. For example, we almost always ignore possible scenarios in which the laws of nature or the geography of the Earth are different. If there were no ocean between Auckland and Sydney, and someone had put a railway line there, one certainly could travel from Auckland to Sydney by train. So it isn't *absolutely* or *logically* impossible to travel from Auckland to Sydney by train. Nor is it absolutely impossible to travel the 2,200 km route in 1 millisecond, even though that contradicts our laws of physics. Something is absolutely (logically, unrestrictedly) impossible only if there is no way it could have been the case, not ignoring any way things could have been. For example, it is absolutely impossible to travel the 2,200 km from Auckland to Sydney in 1 millisecond while travelling at an average speed of 100 km/h.

Some philosophers hold that metaphysical modality and absolute modality coincide: something is absolutely possible/impossible just in case it is metaphysically possible/impossible. Others hold that metaphysical possibility is more restricted than absolute possibility. Yet others hold that there is no such thing as metaphysical modality, or no such thing as absolute modality. We won't enter into these debates.

From a logical perspective, the idea of absolute modality is attractive because the logic of absolute modality is especially simple and well-behaved. As we will see in the next chapter, if we interpret the box and the diamond as absolute necessity and possibility, then the schemas **T**, **D**, **K**, **4**, **5**, and **G** all plausibly become valid, and there is a simple method for checking whether any schema, no matter how complex, is valid.

Exercise 1.8

Which of **T**, **K**, and **4** do you think are valid if we read the box as 'it is obligatory that' and the diamond as 'it is permitted that'?

Exercise 1.9

Translate the following sentences, as well as possible, into $\mathfrak{L}_M$, assuming that the diamond expresses epistemic possibility ('it might be that') and the box epistemic necessity ('it must be that').

(a) I may have offended the principal.
(b) It can't be raining.
(c) Perhaps there is life on Mars.
(d) If the murderer escaped through the window, there must be traces on the ground.

Exercise 1.10

Translate the following sentences, as well as possible, into $\mathfrak{L}_M$, assuming that the diamond expresses deontic possibility ('it is permitted that') and the box deontic necessity ('it is obligatory that').

(a) I must go home.
(b) You don't have to come.
(c) You can't have another beer.
(d) If you don't have a ticket, you must pay a fine.
(e) You need a special visa to enter Chukotka.

Exercise 1.11

Translate the following sentences, as well as possible, into $\mathfrak{L}_M$, assuming that the diamond expresses (some relevant sub-flavour of) circumstantial possibility and the box circumstantial necessity.

(a) It could have snowed today.
(b) It's impossible for me to both cook and entertain the children.
(c) I can't hear you if you're talking to me from the kitchen.
(d) If you can't go to the station, you can't take the train.

1.5 Beyond modality

Despite its name, modal logic extends well beyond the study of modality.

In chapter 7 we will look at a branch of modal logic used to reason about the flow of time, called *temporal logic*. Here we will have operators that function like 'it was the case that' and 'it will be the case that'.

In chapter 8, we will turn to *conditional logic*. Here we will introduce several (non-truth-functional) two-place operators and investigate to what extent they match certain 'if . . . then . . . ' constructions in English.

In chapter 4, we will briefly look at *provability logic*, which studies formal properties of mathematical provability. Here the box is read as 'it is mathematically provable that'.

There are many other branches of modal logic that we won't be able to cover. For example, one lively branch is *dynamic logic*, which is used to reason about how certain actions or events change the state of some system. For any relevant action x, dynamic logic introduces a box-like operator $[x]$ and a diamond-like operator $\langle x \rangle$; $[x]A$ means that the action x will definitely lead to outcome A, while $\langle x \rangle A$ means that x may lead to A. Dynamic logic is widely used in computer science, where the relevant "actions" are typically steps of a computer program.

The many branches of modal logic aren't isolated disciplines. There is a reason why they are commonly treated under the unifying heading of modal logic. Many tools and techniques that have been developed for one branch can also be used for others, and many of the applications share a common abstract core.

What's more, the different branches are often combined. For example, *dynamic epistemic logic* combines ideas from dynamic and epistemic logic to model how knowledge and information change across time. (We will covertly do a bit of dynamic epistemic logic in chapter 5.) It can also be useful to combine, say, deontic and epistemic logic, to reason about what people know about their obligations, or deontic and alethic/circumstancial logic, perhaps to scrutinize the idea that 'ought' implies 'can'.

To conclude this introductory chapter, let's have a quick look at a fun little application that connects to a topic we have discussed.

Suppose we add to the language of propositional modal logic a sentence operator $\Box$ so that $\Box A$ means that A is logically true – true in virtue of its logical form. On this interpretation, $\Box(p \lor \neg p)$ is true, because $p \lor \neg p$ is true in virtue of its form; $\Box(p \lor q)$ is false, because $p \lor q$ is not true in virtue of its form.

Let's revisit the schemas we have considered in section 1.2, beginning with **K**.

$$\Box(A \to B) \to (\Box A \to \Box B) \tag{\textbf{K}}$$

If a conditional $A \to B$ is true in virtue of its form, and so is A, can we conclude that B is true in virtue of its form? Arguably yes. Intuitively, if A is true in any conceivable scenario under any interpretation of the sentence letters, and $A \to B$ is true in any conceivable scenario under any interpretation of the sentence letters, then so is B – for B is bound to be true in any scenario in which A and $A \to B$ are both true.

T is easier.

$$\Box A \to A \tag{\textbf{T}}$$

If something is logically true, then we can surely infer (as a matter of logic) that it is true.

What about **4**?

$$\Box A \to \Box\Box A \tag{\textbf{4}}$$

Take an example. $p \lor \neg p$ is logically true; so $\Box(p \lor \neg p)$ is true. You don't need to know what p means in order to see that $\Box(p \lor \neg p)$ is true, nor do you need to know any substantive facts about the world: the statement is true in virtue of its logical form. So $\Box\Box(p \lor \neg p)$ is true as well. In general, if A is true in virtue of its form, then $\Box A$ is also true in virtue of its form. So $\Box A$ does entail $\Box\Box A$.

Next, **5**:

$$\Diamond A \to \Box\Diamond A \tag{5}$$

I haven't introduced a diamond symbol yet; let's stipulate that $\Diamond A$ is the dual of $\Box A$. So $\Diamond A$ means that $\neg A$ is *not* true in virtue of its logical form. **Dual1** and **Dual2** are then trivially valid. **5** says that if $\neg A$ is not true in virtue of its form, then it is true in virtue its form that $\neg A$ is not true in virtue of its form. This isn't easy to understand, but the following line of thought shows that it is plausible.

Suppose the antecedent of **5** is true: $\neg A$ is not true in virtue of its form. If a sentence is not true in virtue of its form, then evidently any sentence of the same form also isn't true in virtue of its form. So, given that $\neg A$ is not true in virtue of its form, one can tell merely by the logical form of $\neg A$ that $\neg A$ is not true in virtue of its form – in other words, that $\Box\neg A$ is false. So one can tell merely by the logical form of $\Box\neg A$ that it is false. And so one can tell merely by the logical form of $\neg\Box\neg A$ – equivalently, $\Diamond A$ – that it is true. So from the assumption $\Diamond A$ we can logically infer $\Box\Diamond A$. So **5** is valid.

Exercise 1.12

Explain why **D** and **G** are valid on the present interpretation of the box and the diamond.

Exercise 1.13

(a) Which of the schemas we have considered are valid if the box is interpreted as 'it is true that' (and the diamond as the dual of the box)?

(b) Which of the schemas are valid if the box is interpreted as 'it is either true or false that' (and the diamond as the dual)?