# 3 Accessibility

## 3.1 Variable modality

In the last chapter, we assumed that the box and the diamond quantify unrestrictedly over all possible worlds in a model. This has the consequence that modal sentences do not change their truth-value from world to world: if a sentence of the form $\Box A$ (or $\Diamond A$) is true at some world in a model, then it is automatically true at all worlds in the model.

For some interpretations of the box and the diamond, this is fine, but for many others, it is not. Suppose we read the box as ‘it is known that’. Whether something is known varies from world to world. In some worlds, it is known who murdered Richard Montague, in others it is not. Or suppose we read the box as ‘obligatory’. Again, what is obligatory plausibly varies from world to world. In worlds where you have promised to cook dinner, you are under an obligation to cook dinner; in other worlds, you do not have that obligation.

More obviously, consider temporal logic, where the box is read as ‘from now on, it is always going to be the case that’. In models of temporal logic, the “worlds” $W$ are interpreted as times; $\Box p$ is true at $t$ iff $p$ is true at all times *after* $t$. So the box does not quantify unrestrictedly over all times.

In this chapter, we will generalise the semantics from the previous chapter to allow for applications like these. The generalisation is easy. Intuitively, we assume that for any world $w$ in $W$, there is a set of worlds that are possible *relative to* $w$; $\Box p$ is true at $w$ iff $p$ is true at all worlds that are possible relative to $w$. If a world $v$ is possible relative to $w$ we also say that $v$ is *accessible from* $w$, or that $w$ *can see* $v$.

Accessibility means different things in different applications. In epistemic logic, a world $v$ is accessible from $w$ iff $v$ is compatible with what is known at $w$. In the logic of circumstantial modality, $v$ is accessible from $w$ iff $v$ is compatible with the relevant circumstances at $w$. And so on.

In general, a model for $\mathfrak{L}_M$ now has to specify for any pair of worlds whether the first is accessible from the second. This marks the difference between a "basic model" and a "Kripke model".

**Definition 3.1**

A **Kripke model** of $\mathfrak{L}_M$ is a triple $\langle W, R, V\rangle$ consisting of

- a non-empty set $W$,
- a binary relation $R$ on $W$, and
- a function $V$ that assigns to each sentence letter of $\mathfrak{L}_M$ and each member of $W$ a truth-value.

$R$ is the accessibility relation. It is called a relation "on $W$" because it holds between members of $W$.

We also need to update definition 2.2, which settles under which conditions an $\mathfrak{L}_M$-sentence is true at a world in a model. The old definition had the following clauses for the box and the diamond:

(g) $M, w \models \Box A$ iff $M, v \models A$ for all $v \in W$.
(h) $M, w \models \Diamond A$ iff $M, v \models A$ for some $v \in W$.

In the new semantics, the box and the diamond only quantify over accessible worlds:

(g) $M, w \models \Box A$ iff $M, v \models A$ for all $v \in W$ such that $wRv$.
(h) $M, w \models \Diamond A$ iff $M, v \models A$ for some $v \in W$ such that $wRv$.

Here is the full definition, for completeness.

**Definition 3.2: Kripke Semantics**

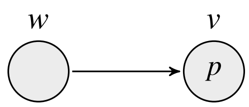
If $M = \langle W, R, V\rangle$ is a Kripke model, $w$ is a member of $W$, $\rho$ is any sentence letter, and $A$, $B$ are any $\mathfrak{L}_M$-sentences, then

| | | |
|---|---|---|
| (a) | $M, w \models \rho$ | iff $V(\rho, w) = 1$. |
| (b) | $M, w \models \neg A$ | iff $M, w \not\models A$. |
| (c) | $M, w \models A \wedge B$ | iff $M, w \models A$ and $M, w \models B$. |
| (d) | $M, w \models A \vee B$ | iff $M, w \models A$ or $M, w \models B$. |
| (e) | $M, w \models A \rightarrow B$ | iff $M, w \models B$ or $M, w \not\models A$. |
| (f) | $M, w \models A \leftrightarrow B$ | iff $M, w \models (A \rightarrow B)$ and $M, w \models (B \rightarrow A)$. |
| (g) | $M, w \models \Box A$ | iff $M, v \models A$ for all $v \in W$ such that $wRv$. |
| (h) | $M, w \models \Diamond A$ | iff $M, v \models A$ for some $v \in W$ such that $wRv$. |

When I speak of truth at a world in a Kripke model, this should always be understood in accordance with definition 3.2. Definition 2.2 defines truth at a world in a basic model.

Like definition 2.2, definition 3.2 settles the truth-value of any $\mathfrak{L}_M$-sentence at any world in any model. Let's go through a few examples.

Consider a model with two worlds, $w$ and $v$; $v$ is accessible from $w$, and no world is accessible from $v$; the interpretation function assigns 1 to $p$ at $v$ and 0 to all other sentence letters and worlds. The model can be pictured like this, with an arrow representing accessibility:




With the help of definition 3.2, we can figure which $\mathfrak{L}_M$-sentences are true at which worlds in the model. For example:

- By clause (a) of definition 3.2, $p$ is true at $v$ and false at $w$.
- By clause (g), $\Box p$ is true at $w$, because $p$ is true at $v$ and $v$ is the only world accessible from $w$.
- By clause (h), $\Diamond p$ is true at $w$, because $p$ is true at $v$ and $v$ is accessible from $w$.
- By clause (h), $\Diamond p$ is false at $v$, because there is no world accessible from $v$ at which $p$ is true.
- By clause (g), $\Box\Diamond p$ is false at $w$, because $\Diamond p$ is false at $v$ and $v$ is accessible from $w$.

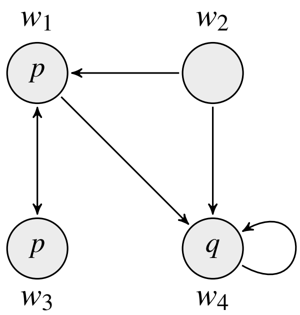
- By clause (g), $\Box\Diamond p$ is true at $v$, because there is no world accessible from $v$ at which $\Diamond p$ is false.

Note that $\Diamond p$ and $\Box\Diamond p$ do not always have the same truth-value. In the new semantics, we can no longer ignore all but the last operator in a string of operators. Note also that $\Box p$ is true at $w$ even though $p$ is false; so $\Box p \to p$ is no longer valid.

**Exercise 3.1**

Explain why any sentence of the form $\Box A$ is true at world $v$ in the above model.

Another example:




Using definition 3.2, we can figure out the following (among other things).

- $\Box p$ is false at $w_1$, because $w_1$ can see $w_4$, where $p$ is false.
- $\Box p$ is true at $w_3$, because $w_3$ can only see $w_1$, where $p$ is true.
- $\Diamond\Box p$ is true at $w_1$, because $w_1$ can see $w_3$ and $\Box p$ is true at $w_3$.
- $\Diamond q$ is true at $w_4$, because $w_4$ can see itself, and $q$ is true at $w_4$.
- $\Diamond\Diamond q$ is true at $w_1$, because $w_1$ can see $w_4$, and $\Diamond q$ is true at $w_4$.

The next three exercises all refer to this example.

**Exercise 3.2**

At which worlds in the model are the following sentences true?

(a) $p \vee \neg q$
(b) $\Box(p \vee \neg q)$
(c) $\Diamond(\neg p \wedge \neg q)$

(d) $\Diamond\Box q$
(e) $\Diamond\Diamond\Box q$

**Exercise 3.3**

For each world in the model, find an $\mathfrak{L}_M$-sentence that is true only at that world.

**Exercise 3.4**

Can you draw a diagram of a smaller model (with fewer worlds) in which the exact same $\mathfrak{L}_M$-sentences are true at $w_1$?

## 3.2 The systems K and S5

Remember that a sentence is *valid* iff it is true at all worlds in all models. Different conceptions of a model, and of what it means for a sentence to be true at a world in a model, give rise to different kinds of validity. For example, $\Box p \to p$ is valid by the definitions from the previous chapter, but not by our present definitions.

To avoid confusion, it is best to use different expressions for the different kinds of validity. Let's call the new kind of validity *K-validity*. The old kind will henceforth be called *S5-validity*, because the sentences that are valid by the definition from the previous chapter are precisely the sentences contained in C.I. Lewis's system S5.

**Definition 3.3**

A sentence $A$ is **K-valid** (for short, $\models_K A$) iff $A$ is true at every world in every Kripke model.

The same distinction applies to the concept of entailment or logical consequence. Logical consequence in the old sense (definition 2.4) will henceforth be called *S5-consequence*. The new sense is that of *K-consequence*.

**Definition 3.4**

A sentence $A$ is a **K-consequence** of a set $\Gamma$ of sentences (for short, $\Gamma \models_K A$) iff $A$ is true at any world in any Kripke model at which all members of $\Gamma$ are true.

As before I'll also apply these notions to schematic sentences. For example, a schema is K-valid iff all its instances are K-valid.

Many properties of S5-consequence and S5-validity carry over to K-consequence and K-validity. In particular, observation 1.1 (p. 11), the propositional extension theorem (p. 33), and the replacement theorem (p. 34) still hold, and for the same reasons as before. I won't go through the arguments again.

I already mentioned that the set of all S5-valid sentences is known as **system S5**. The set of all K-valid sentences is known as **system K**. Intuitively, these sets are called "systems" because they aren't random collection of sentences; the sentences in K or S5 are systematically related to one another. Systems are also often called **logics**.

You might think a logic should do more than identify a class of valid sentences: it should also tell us which sentences follow from which others. But our systems actually do that, albeit in an indirect manner. By observation 1.1, we can convert questions about logical consequence into questions about validity. For example, instead of asking whether $\Box p$ entails $p$, we can equivalently ask whether $\Box p \to p$ is valid. The systems S5 and K answer all questions about validity, and so they indirectly settle which sentences entail which others. In S5, $\Box p$ entails $p$, in K it does not.

From the previous chapter, we know that S5 contains

- all instances of the **K**-schema $\Box(A \to B) \to (\Box A \to \Box B)$,
- all instances of the **T**-schema $\Box A \to A$,
- all instances of the **D**-schema $\Box A \to \Diamond A$,
- all instances of the **4**-schema $\Box A \to \Box\Box A$,
- all instances of the **5**-schema $\Diamond A \to \Box\Diamond A$, and
- all instances of the **G**-schema $\Diamond\Box A \to \Box\Diamond A$.

Which of these do we have in system K?

As we saw above, we do not have all instances of **T**, for there are worlds in some Kripke models at which $\Box p \to p$ is false. So the **T** schema is not K-valid. Nor are the schemas **D**, **4**, **5**, and **G**.

**Exercise 3.5**

Give a countermodel to **D**. That is, define a Kripke model in which $\Box p \to \Diamond p$ is false at some world $w$.

**Exercise 3.6**

Can you find an instance of the **T** schema that is K-valid?

The **K** schema, however, *is* K-valid, as its name suggests.

**Observation 3.1:** Every instance of **K** is true at every world in every Kripke model.

*Proof:* Let $w$ be an arbitrary world in an arbitrary Kripke model. By clause (e) of definition 3.2, an instance of $\Box(A \to B) \to (\Box A \to \Box B)$ is false at $w$ only if $\Box(A \to B)$ and $\Box A$ are both true at $w$ while $\Box B$ is false. By clause (g) of definition 3.2, $\Box B$ is false at $w$ only if $B$ is false at some world accessible from $w$. But if $\Box(A \to B)$ and $\Box A$ are both true at $w$, $A \to B$ and $A$ would have to be true at any such world, again by clause (g). But there can be no world at which $A \to B$ and $A$ are true while $B$ is false, by clause (e) of definition 3.2. □

**Exercise 3.7**

Show that $\Box(A \lor \neg A)$ is K-valid.

We have found sentences that are in S5 but not in K. Are there also sentences in K that are not in S5? You may want to pause and think about this for a moment.

**Observation 3.2:** Every K-valid sentence is S5-valid.

*Proof:* In essence, observation 3.2 holds because the basic models from the previous chapter can be simulated by Kripke models in which all worlds have universal access to all worlds. If a sentence is K-valid and hence true in every Kripke model, it must also true in every "universal" Kripke model; these models behave just like basic models; so the sentence is also true in every basic model.

It is worth going through this more carefully. For any basic model $M = \langle W, V \rangle$, let $M^*$ be the Kripke model $\langle W, R, V \rangle$ with the same worlds $W$ and the same interpretation function $V$, and with an accessibility relation $R$ that holds between all worlds in $W$; so every world in $M^*$ can see every other world as well as itself. If every world can see every world, then it makes no difference whether we use definition 2.2 or definition 3.2 to evaluate the truth of sentences at a world. That's because the two definitions only differ for the case of the modal operators, which definition 2.2 interprets as quantifiers over all worlds, while definition 3.2 interprets them as quantifiers over the accessible worlds. So we have:

(*) A sentence is true at a world $w$ in a basic model $M$ iff it is true at $w$ in the corresponding Kripke model $M^*$.

(A full proof of (*) would proceed by induction on complexity of the sentence.) Now suppose a sentence $A$ is *not* S5-valid, meaning that it is false at some world $w$ in some basic model $M$. By (*), it follows that $A$ is also false at some world in some Kripke model (namely, in $M^*$). And if $A$ is false at some world in some Kripke model, then $A$ is not K-valid. By contraposition, this means that if $A$ is K-valid, then $A$ is S5-valid. □

## 3.3 Some other normal systems

For many applications of modal logic, we need a concept of validity that lies in between K-validity and S5-validity. For example, suppose we read the box as physical necessity and the diamond as physical possibility, understood as compatibility with the laws of nature. On a certain conception of the laws of nature, the laws at a world cannot be violated at that world: anything that happens must be compatible with the laws. Equivalently, anything that is physically necessary is actually the case. In the logic of physical necessity, $\Box A$ should therefore entail $A$. On the other hand, it is not clear if $\Box A$ should entail $\Box\Box A$: if $A$ is physically necessary, can we infer that it is physically necessary that $A$ is physically necessary? Some have argued that we can't.

If that is right, then the logic of physical necessity lies in between K and S5. We want $\Box A \to A$ to be valid, but not $\Box A \to \Box\Box A$. System K gives us neither schema, S5 gives us both.

Fortunately, it is easy to define systems in between K and S5, by putting restrictions on the accessibility relation in Kripke models.

Let's say that an $\mathfrak{L}_M$-sentence is **valid in a class of Kripke models** iff the sentence is true at every world in every model in the class. (A *schema* is valid in a class of models iff all its instances are valid in the class.)

K-validity is validity in the class of all Kripke models. $\Box p \to p$ is not K-valid, because it is false at certain worlds in certain Kripke models. All these worlds have something in common: they do not have access to themselves. If we rule out models in which some worlds are inaccessible from themselves, the **T**-schema becomes valid.

**Observation 3.3:** The **T**-schema is valid in the class of Kripke models in which every world is accessible from itself.

*Proof:* According to clause (e) of definition 3.2, $\Box A \to A$ is false at a world $w$ only if $\Box A$ is true at $w$ and $A$ is false; but if $\Box A$ is true at $w$ and $w$ has access to itself, then by clause (g) of definition 3.2, $A$ is true at $w$. So if $\Box A \to A$ is false at $w$, and $w$ is accessible from itself, then $A$ is both true and false at $w$, which is impossible. Hence $\Box A \to A$ is true at every world in every model in which every world is accessible from itself. □

A relation $R$ on a set $W$ is called **reflexive** if each member of $W$ is $R$-related to itself. If the accessibility relation in a Kripke model is reflexive, we'll also call the model itself reflexive. So observation 3.3 states that the **T**-schema is valid in the class of reflexive Kripke models.

The set of all sentences valid in the class of reflexive Kripke models is known as **system T**. System T is *stronger* than K, in the sense that it contains sentences not in K, but it is *weaker* than S5: it does not contain all of S5.

Systems of modal logic often share their name with a particular schema. To avoid confusion, I generally use bold-face letters for schemas and normal-face letters for systems. K and T are systems, **K** and **T** are schemas. All instances of **T** are in T, but many sentences in T (for example, all instances of **K**) are not instances of **T**.

Above I mentioned a temporal application of modal logic, in which the box is read as 'it is always going to be the case that'. Here, $\Box p$ should count as true at a given time $t$ iff $p$ is true at all times *after* $t$. In the relevant Kripke models, the accessibility relation $R$ is the earlier-later relation between times: $t_1 R t_2$ iff $t_1$ is earlier than $t_2$. On that interpretation, we don't want to assume that $R$ is reflexive, which would mean that every point in time is earlier than itself. But we'll want something else. Suppose $t_1$ is earlier than $t_2$, and $t_2$ is earlier than $t_3$. Then surely $t_1$ is earlier than $t_3$. That is, we should restrict the relevant models to ones in which the accessibility relation is transitive.

A relation $R$ is called **transitive** if, whenever $xRy$ and $yRz$ then $xRz$. Again, we will call a Kripke model transitive if its accessibility relation is transitive.

The set of sentences that are valid in the class of transitive Kripke models is known as **system K4**. The label alludes to the fact that (a) K4 is an extension of K, and (b) the **4**-schema is K4-valid. (a) is obvious. Here is the proof of (b):

**Observation 3.4:** The **4**-schema is valid in the class of transitive Kripke models.

*Proof:* Suppose for reductio that there is some transitive Kripke model in which some instance of $\Box A \to \Box\Box A$ is false at some world $w$. By clause (e) of definition 3.2, it follows that (1) $\Box A$ is true at $w$ and (2) $\Box\Box A$ is false at $w$. By clause (g) of definition 3.2, (2) implies that there is some world $v$ accessible from $w$ where $\Box A$ is false. And that, in turn implies that there is some world $u$ accessible from $v$ at which $A$ is false. Since $R$ is transitive, $u$ is accessible from $w$. By (1), $A$ is true at $u$. So $A$ is both true and false at $u$. Contradiction. □

We can combine the systems T and K4 by requiring both reflexivity and transitivity. The set of sentences valid in the class of reflexive and transitive Kripke models is C.I. Lewis's **system S4**. Both **T** and **4** are S4-valid.

There are many other conditions we could impose on the accessibility relation, and many combinations of these conditions. Each of these defines a system of modal logic. Here are some well-known model classes with the conventional names for the corresponding systems, repeating (for future reference) the ones we already know.

| *System* | *Constraint on R* |
|---|---|
| K | – |
| T | $R$ is **reflexive**: every world in $W$ can access itself |
| D | $R$ is **serial**: every world in $W$ can access some world |
| K4 | $R$ is **transitive**: whenever $wRv$ and $vRu$, then $wRu$ |
| K5 | $R$ is **euclidean**: whenever $wRv$ and $wRu$, then $vRu$ |
| B | $R$ is reflexive and **symmetric**: whenever $wRv$ then $vRw$ |
| S4 | $R$ is reflexive and transitive |
| S4.2 | $R$ is reflexive, transitive, and **convergent**: whenever $wRv$ and $wRu$, then there is some $t$ such that $vRt$ and $uRt$ |
| S5 | $R$ is reflexive, transitive, and symmetric |
| S5 | $R$ is **universal**: every world has access to every world |

We will have a closer look at some of these systems in later chapters, when we turn to applications of modal logic.

S5 occurs twice in the above list. We already know S5 as the system for universal models, in which the box and the diamond quantify unrestrictedly over the whole space $W$. But we also get S5 if we require the accessibility relation to be reflexive, transitive, and symmetric.

Relations that are reflexive, transitive, and symmetric are called **equivalence relations**. An equivalence relation on a set $W$ divides the members of $W$ into classes within which everyone stands in the relation to everyone. (These classes are called **equivalence classes**.)

For example, let $S$ be the relation that holds between two people if they have the same birthday. This is an equivalence relation: it is reflexive (everyone has the same birthday as themselves), transitive (if $aSb$ and $bSc$ than $aSc$), and symmetric (if $aSb$ then $bSa$). for any person $a$, consider the class $[a]_S$ of everyone who has the same birthday as $a$. Everyone in $[a]_S$ has the same birthday as everyone else in $[a]_S$. So within $[a]_S$, the same-birthday relation $S$ is universal.

Now let me explain why the above two characterisations of S5 are equivalent.

**Observation 3.5:** A sentence is valid in the class of Kripke models whose accessibility relation is universal iff it is valid in the class of Kripke models whose accessibility relation is an equivalence relation.

*Proof:* The right-to-left direction is easy. If $R$ is the universal relation on $W$, then $R$ is reflexive, transitive, and symmetric. So the universal relations are a special kind of equivalence relation. If a sentence is valid in every model in which $R$ is an equivalence relation, it must therefore be valid in every model in which $R$ is universal.

The other direction is more interesting. We argue by contraposition, showing that if a sentence $A$ is not valid in the class of models in which $R$ is an equivalence relation, then $R$ is also not valid in the class of universal models. So suppose $A$ is not valid in the class of models in which $R$ is an equivalence relation. Then there is some world $w$ in some such model $M = \langle W, R, V \rangle$ such that $M, w \not\models A$. Define the new model $M' = \langle W', R', V' \rangle$ as follows:

> $W'$ is the class of worlds accessible in $M$ from $w$ (i.e., the equivalence class $[w]_R$).
>
> $R'$ is the universal relation on $W'$.
>
> $V'$ assigns the same truth-value as $V$ to all sentence letters within $W'$.

$M'$ has a universal accessibility relation. But from the perspective of $w$, $M$ and $M'$ are indistinguishable. What (if anything) happens outside of $[w]_R$ in $M$ makes no difference to the truth-value of any sentence at $w$: any sentence is true at $w$ in $M$ iff it is true at $w$ in $M'$. This could be shown by induction on complexity, but I hope you see intuitively why it is the case.

From the assumption that $A$ is false at some model whose accessibility relation is an equivalence relation, we can therefore infer that $A$ is false in some model whose accessibility relation is universal. □

**Exercise 3.8**

Let $R$ be the relation on the set of people that holds between $x$ and $y$ iff $y$ is at least as old as $x$. Is $R$ reflexive? serial? transitive? euclidean? symmetric?

universal?

**Exercise 3.9**

Explain these facts:

(a) If $R$ is symmetric and transitive, then $R$ is euclidean.
(b) If $R$ is symmetric and euclidean, then $R$ is transitive.
(c) If $R$ is reflexive and euclidean, then $R$ is symmetric.

**Exercise 3.10**

What is wrong with the following argument? "If $R$ is symmetric, then $wRv$ implies $vRw$; if $R$ is transitive, it follows that $wRw$. So symmetry and transitivity together imply reflexivity."

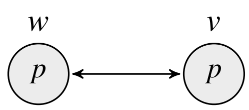
## 3.4 Frames

There is a close connection between conditions on the accessibility relation in Kripke models and modal schemas: between reflexivity and the **T** schema, between transitivity and the **4** schema, and so on. But what exactly is that connection?

You might think it's this:

(?) **T** is valid in a class of models iff all models in the class are reflexive;
**4** is valid in a class of models iff all models in the class are transitive,
and so on.

But that's false. Take the case of **T** and reflexivity. We know (observation 3.3) that **T** is valid in the class of reflexive models. It follows that if all models in a class are reflexive, then **T** is valid in that class. But the other direction fails. For a counterexample, consider the following model.




There are two worlds, both of which can see each other; neither can see itself. $p$ is true at both worlds, all other sentence letters are false at both worlds. This model is not reflexive, but no instance of the **T** schema $\Box A \to A$ is false at any world in the model. (Try to find a false instance!) The fact that the **T** schema is valid in a class of models therefore does not entail that all models in the class are reflexive, for the class might contain models like the one just described.

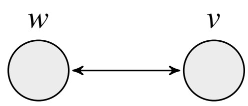
To understand the connection between modal schemas and conditions on the accessibility relation, we need to talk about *frames*. A frame is what you get if take a Kripke model and remove the interpretation function.

**Definition 3.5**

A **frame** is a pair of a set $W$ and a relation $R$ on $W$.

Roughly speaking, if we think of a model as representing a scenario and an interpretation, then a frame is the part of the model that represents the scenario. For modal logic, we only care about the abstract structure of a scenario: how many worlds there are, and how they are accessible from one another.

Frames can be pictured just like Kripke models, but without any sentence letters in the nodes. The frame of the above model looks like this.




Now remember that validity is supposed to mean truth in virtue of the meaning of the logical expressions. Whether a sentence is valid should therefore not depend on the meaning of the non-logical expressions. So if we define a particular kind of validity by reference to a class of Kripke models, the constraints we impose on the models in the class should be constraints on the frame of the models, not on the interpretation function.

For example, suppose I suggested that a sentence is "$X$-valid" iff it is true at all worlds in all Kripke model whose interpretation function assigns 1 to the sentence letter $p$ at every world. Then $\Box p$ comes out $X$-valid, while $\Box q$ is $X$-invalid. Intuitively, $X$-validity is not a sensible concept of validity because $\Box p$ and $\Box q$ have the same logical form. If $\Box p$ is true in virtue of the meaning of the box, then $\Box q$ should also be true in virtue of the meaning of the box. The systems from the previous section were all defined reasonably, by putting constraints on the frame of a Kripke model, rather than the interpretation function.

Let's say that a sentence is **valid on a frame** if it is true at all worlds in all models with that frame. A sentence is **valid in a class of frames** if it valid on all frames in the class.

Evidently, if a sentence is valid in the class of all models whose accessibility relation satisfies a certain condition, then it is also valid in the class of all frames whose accessibility relation satisfies that condition, and vice versa. So we could equivalently define system **T**, for example, as the set of sentences valid in the class of reflexive frames; **K4** is the set of sentences valid in the class of transitive frames; and so on. (A reflexive/transitive/etc. frame is a frame with a reflexive/transitive/etc. accessibility relation.)

Now here is the connection between **T** and reflexivity: **T** is valid in a class of frames iff all frames in the class are reflexive. More simply:

**Observation 3.6:** **T** is valid on a frame iff the frame is reflexive.

*Proof:* The right-to-left direction follows from observation 3.3, according to which **T** is valid in the class of reflexive models, and therefore in the class of reflexive frames, and therefore on any frame in that class. For the other direction, we have to show that if (all instances of) **T** are valid on a frame $\langle W, R \rangle$, then $R$ is reflexive. We do this by showing that if $R$ is not reflexive, then we can find an interpretation function $V$ that makes $\Box p \to p$ false at some world $w$. $w$ will be an arbitrary world in $W$ that can't see itself. (There must be some such world if $R$ is not reflexive.) We let $V$ assign 0 to $p$ at $w$ and 1 to $p$ at all other worlds. Then $\Box p$ is true at $w$ and $p$ false, and so $\Box p \to p$ is false at $w$. □

If a schema is valid on all and only the frames whose accessibility relation satisfies a certain property, the schema is said to **correspond** to that property. Observation 3.6 therefore says that the **T** schema corresponds to reflexivity.

Instead of proving more facts about the correspondence between modal schemas and frame conditions, I will simply give you a list of some important results.

| *Schema* | | *Corresponding Frame Condition* |
|---|---|---|
| **T** | $\Box A \to A$ | $R$ is reflexive: every world in $W$ is accessible from itself |
| **D** | $\Box A \to \Diamond A$ | $R$ is serial: every world in $W$ can access some world in $W$ |
| **B** | $A \to \Box\Diamond A$ | $R$ is symmetric: whenever $wRv$ then $vRw$ |
| **4** | $\Box A \to \Box\Box A$ | $R$ is transitive: whenever $wRv$ and $vRu$, then $wRu$ |
| **5** | $\Diamond A \to \Box\Diamond A$ | $R$ is euclidean: whenever $wRv$ and $wRu$, then $vRu$ |
| **G** | $\Diamond\Box A \to \Box\Diamond A$ | $R$ is convergent: whenever $wRv$ and $wRu$, then there is some $t$ such that $vRt$ and $uRt$ |

Correspondence facts are often useful when trying to figure out which schemas should be valid on a given interpretation of the modal operators. Return to the case of physical possibility and necessity. Above I asked whether the **4** schema $\Box A \to \Box\Box A$ should count as valid on this interpretation. The question is hard to answer by direct intuition: if something is physically necessary, is it physically necessary that it is physically necessary? Since the **4** schema corresponds to transitivity, we can equivalently ask whether the physical accessibility relation is transitive. That is, if a world $v$ is physically possible relative to a world $w$, and $u$ is physically possible relative to $v$, is $u$ always physically possible relative to $w$?

The answer depends (among other things) on how we understand physical possibility. Earlier I suggested that a world $v$ is physically possible relative to a world $w$ if nothing that happens at $v$ contradicts the laws of nature at $w$. This does not imply that $v$ has the same laws as $w$. To illustrate, suppose the only law at $w$ is that ravens are black; at $v$, there is no such law but there happen to be no non-black ravens. Then what happens at $v$ does not contradict the laws at $w$, even though $v$ has different laws. Relative to the laws of $v$, worlds with white ravens are physically possible. So a world accessible from a world that is accessible from $w$ need not itself be accessible from $w$. So **4** is not valid in the logic of physical necessity.

**Exercise 3.11**

Can you find frame conditions that correspond to these schemas?

(a) $\Box A \leftrightarrow A$

(b) $\Box A$

## 3.5 More trees

In section 2.5, I described the tree method for checking whether a sentence or schema is valid, and for constructing countermodels. These were the rules for the box and the diamond:

| $\Box A \quad (\omega)$ | $\Diamond A \quad (\omega)$ | $\neg\Box A \quad (\omega)$ | $\neg\Diamond A \quad (\omega)$ |
|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ |
| $A \quad (v)$ | $A \quad (v)$ | $\neg A \quad (v)$ | $\neg A \quad (v)$ |
| ↑ | ↑ | ↑ | ↑ |
| old | new | new | old |

The rule for $\Box A$ allows us to infer, from the hypothesis that $\Box A$ is true at some world, that $A$ is true at any world that occurs on a tree branch. This made sense given the semantics of the previous chapter, where the box quantified unrestrictedly over all worlds. With the new semantics of the present chapter, we need to change the rules.

If $\Box A$ is true at a world $w$, and there's some other world $v$ on the branch, we can only infer that $A$ is true at $v$ if $v$ is accessible from $w$. So we need to keep track of which worlds are accessible from any world on a tree. We do this by adding meta-linguistic statements about accessibility to the tree.

For example, suppose we want to expand the following node.

n. $\quad \Diamond p \quad (w)$

The node represents the hypothesis that $\Diamond p$ is true at $w$. It follows that $p$ is true at some world $v$. Moreover, that world $v$ must be accessible from $w$. So we add two new nodes:

m. $\quad wRv$

m+1. $\quad p \quad (v)$

Node $m + 1$ is what we would have added by the old rules. Node $m$ is a meta-linguistic statement reminding us that $v$ is accessible from $w$. '$wRv$' is not a sentence of $\mathfrak{L}_M$; it isn't true or false relative to a world, which is why node $m$ has no world label.

What if we want to expand a box node?

n. $\Box p$ (w)

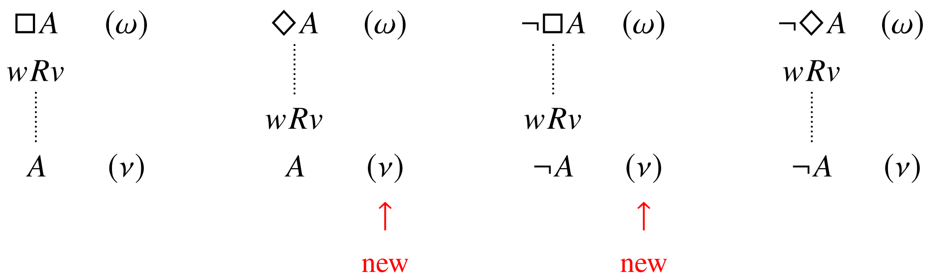
By itself, this doesn't tell us anything about the truth-value of $p$ at any world. We can't infer that $p$ is true at $w$, because $w$ might not be accessible from itself. Indeed, if no world is accessible from $w$, then $\Box p$ can be true even if $p$ is false at every world. So we can't even infer that there is some world or other at which $p$ is true.

However, suppose a branch that contains node $n$ also contains the following node.

m. $wRv$

Now we can infer that $p$ is true at $v$. So to expand a box node on a branch, there must be another node on the branch telling us that the world at which the boxed sentence is true has access to some other world.

Here are diagrams of the new rules for the box and the diamond.




If two nodes occur above the dotted line in a rule, as in the rule for $\Box A$, this means that the rule can only be applied if both nodes already occur on the relevant branch (in any order, and not necessarily adjacent to each other).

The rules for negated boxes and diamonds are as you would expect from the duality of the box and the diamond. Note that only nodes of type $\Diamond A$ and $\neg\Box A$ allow us to introduce hypotheses about accessibility into a tree.

The rule for the classical connectives all stay the same. Together, all these rules are known as the **K-rules**; the tree rules from section 2.5 are the **S5-rules**.

Here is a schematic tree proof to show that $\models_K \Box(A \land B) \to (\Box A \land \Box B)$.

| | | | |
|---|---|---|---|
| 1. | $\neg(\Box(A \wedge B) \to (\Box A \wedge \Box B))$ | $(w)$ | (Ass.) |
| 2. | $\Box(A \wedge B)$ | $(w)$ | (1) |
| 3. | $\neg(\Box A \wedge \Box B)$ | $(w)$ | (1) |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4. | $\neg\Box A$ | $(w)$ | (3) | 5. | $\neg\Box B$ | $(w)$ | (3) |
| 6. | $wRv$ | | (4) | 11. | $wRu$ | | (5) |
| 7. | $\neg A$ | $(v)$ | (4) | 12. | $\neg B$ | $(u)$ | (5) |
| 8. | $A \wedge B$ | $(v)$ | (2,6) | 13. | $A \wedge B$ | $(u)$ | (2,11) |
| 9. | $A$ | $(v)$ | (8) | 14. | $A$ | $(u)$ | (13) |
| 10. | $B$ | $(v)$ | (8) | 15. | $B$ | $(u)$ | (13) |
| | x | | | | x | | |

The annotation '(2,6)' for node 8 indicates that this node is based on two assumptions from earlier in the branch: the assumption on node 2 that $\Box(A \wedge B)$ is true at $w$, and the assumption on node 6 that $wRv$. Only these two assumptions together allow us to infer that $A \wedge B$ is true at $v$.

**Exercise 3.12**

Use the K-rules to check which of the following schemas are K-valid.

(a) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$
(b) $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
(c) $(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B)$
(d) $\Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$
(e) $\Box(A \vee B) \leftrightarrow (\Box A \vee \Box B)$
(f) $\Box(A \to B) \to (\Diamond A \to \Diamond B)$.
(g) $(\Box A \wedge \Diamond B) \to \Diamond(A \wedge B)$.

For systems in between K and S5 that are characterised by certain constraints on the accessibility relation, we add new rules for manipulating accessibility nodes. For example, if we want to check whether a sentence is T-valid, we use a *reflexivity rule* in addition to the K-rules. The reflexivity rule says that if a world variable $\omega$ occurs on a branch, then we may always add $\omega R \omega$ to the branch.

Here is a proof of $\Box p \to p$, using the reflexivity rule.

| | | | |
|---|---|---|---|
| 1. | $\neg(\Box p \to p)$ | $(w)$ | (Ass.) |
| 2. | $\Box p$ | $(w)$ | (1) |
| 3. | $\neg p$ | $(w)$ | (1) |
| 4. | $wRw$ | | (Ref.) |
| 5. | $p$ | $(w)$ | (2,4) |
| | x | | |

To test for validity on a class of transitive frames (or models), we need a *transitivity rule*, which allows us to infer $\omega R \upsilon$ from $\omega R \nu$ and $\nu R \upsilon$. Here is a proof of the **4** schema using this rule.

| | | | |
|---|---|---|---|
| 1. | $\neg(\Box A \to \Box\Box A)$ | $(w)$ | (Ass.) |
| 2. | $\Box A$ | $(w)$ | (1) |
| 3. | $\neg\Box\Box A$ | $(w)$ | (1) |
| 4. | $wRv$ | | (3) |
| 5. | $\neg\Box A$ | $(v)$ | (3) |
| 6. | $vRu$ | | (5) |
| 7. | $\neg A$ | $(u)$ | (5) |
| 8. | $wRu$ | | (4,6,Tr.) |
| 5. | $A$ | $(u)$ | (2,8) |
| | x | | |

The following diagrams summarize the tree rules for the frame conditions we have so far considered.

| Reflexivity | Seriality | Transitivity |
|---|---|---|
| ⋮ | ⋮ | $\omega R \nu$ |
| | | $\nu R \upsilon$ |
| | | ⋮ |
| $\omega R \omega$ | $\omega R \nu$ | $\omega R \upsilon$ |
| ↑ | ↑ ↑ | |
| old | old new | |

| Symmetry | Euclidity | Convergence |
|---|---|---|
| $\omega R \nu$ | $\omega R \nu$ | $\omega R \nu$ |
| ⋮ | $\omega R \upsilon$ | $\omega R \upsilon$ |
| $\nu R \omega$ | ⋮ | ⋮ |
| | $\nu R \upsilon$ | $\nu R \tau$ |
| | | $\upsilon R \tau$ |
| | | ↑ new |

By selectively adding some of these rules to the K-rules, we get tree rules for a variety of modal logics, such as the following. (Compare the table on p. 57.)

| *System* | *Tree Rules* |
|---|---|
| K | K-rules |
| T | K-rules and reflexivity rule |
| D | K-rules and seriality rule |
| K4 | K-rules and transitivity rule |
| K5 | K-rules and euclidity rule |
| B | K-rules and symmetry rule |
| S4 | K-rules, reflexivity rule, and transitivity rule |
| S4.2 | K-rules, reflexivity rule, transitivity rule, and convergence rule |