A. Answers to Selected Exercises

A.1. Chapter 1

Exercise 1.1

An operator O is truth-functional if you can figure out the truth-value of Op from the truth-value of p. So (c) and (g) are truth-functional; (a), (b), (d), and (e) are not truth-functional. (f) is truth-functional if God is omniscient (and infallible); it is also truth-functional if God doesn't exist, or if God believes all and only false things; otherwise (f) is not truth-functional.

Exercise 1.3

(a), (b), and (e).

Exercise 1.4

The following pairs are duals: (a) and (c), (b) and (d), (e) and (g), (f) and (h), (i) and (k), (l) and (l), (m) and (m).

Exercise 1.6

All instances of $\Box\Box A \to \Box A$ are instances of **T**. So if all instances of **T** are valid, then so are all instances of $\Box\Box A \to \Box A$.

Exercise 1.7

I show that if $\Box A \to A$ is valid, then so is $A \to \Diamond A$. The other direction is similar.

$\Box A \to A$ is equivalent to $\neg A \to \neg\Box A$, which (by duality) is equivalent to $\neg A \to \Diamond\neg A$. If $\neg A \to \Diamond\neg A$ is valid then so is $\neg\neg A \to \Diamond\neg\neg A$, because all instances of the latter are instances of the former. And $\neg\neg A \to \Diamond\neg\neg A$ is equivalent to $A \to \Diamond A$.

Exercise 1.12

D is valid because A and $\neg A$ can never both be true in virtue of their form. So if A is true in virtue of its form, then $\neg A$ is not true in virtue of its form.

For **G**, suppose $\Diamond\Box A$. Then there is some scenario at which $\Box A$ is true under some interpretation of the non-logical vocabulary. But whether A is logically true doesn't vary from scenario to scenario or from interpretation to interpretation. So $\Box A$ is true at every scenario under every interpretation (of the non-logical vocabulary). And then $\Diamond A$ is true at every scenario under every interpretation (as reasoned above for **D**). So $\Box\Diamond A$.

Exercise 1.13

(a) All of them.
(b) **K**, **Dual1**, **Dual2**, **4**, **5**, and **G**, but not **T** or **D**.

A.2. Chapter 2

Exercise 2.2

At both worlds.

Exercise 2.3

Suppose for reductio that some instance of $\Box A \to \Diamond A$ is false at some world w in some model. By definition 2.2, $\Box A$ is then true at w and $\Diamond A$ false. But if $\Diamond A$ is false at w then (by definition 2.2) A is false at every world in the model, including w. And then $\Box A$ isn't true at w (by definition 2.2). Contradiction.

Exercise 2.4

5 reduces to the tautology $\Diamond A \to \Diamond A$; **G** reduces to $\Box A \to \Diamond A$, whose validity we proved above.

Exercise 2.5

Suppose A is valid. So A is true at all worlds in all models (by definition 2.3). It follows that in any given model, A is true at every world. By definition 2.2, it follows that $\Box A$ is true at every world in any model.

Exercise 2.8

(a) Target: $p \rightarrow \Box(p \lor q)$

1.	$\neg(p \rightarrow \Box(p \lor q))$	(w)	(A)
2.	p	(w)	(1)
3.	$\neg\Box(p \lor q)$	(w)	(1)
4.	$\neg(p \lor q)$	(v)	(3)
5.	$\neg p$	(v)	(4)
5.	$\neg q$	(v)	(4)

Countermodel: $W = \{w, v\}, V(p, w) = 1, V(p, v) = 0, V(q, v) = 0$.

(b) Target: $\Box p \lor \Box\neg p$

1.	$\neg(\Box p \lor \Box\neg p)$	(w)	(A)
2.	$\neg\Box p$	(w)	(1)
3.	$\neg\Box\neg p$	(w)	(1)
4.	$\neg p$	(v)	(2)
5.	$\neg\neg p$	(u)	(3)
6.	p	(u)	(5)

Countermodel: $W = \{w, v, u\}, V(p, v) = 0, V(p, u) = 1$.

(c) Target: $\Diamond(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)$

1.	$\neg(\Diamond(p \to q) \to (\Diamond p \to \Diamond q))$	(w)	(A)
2.	$\Diamond(p \to q)$	(w)	(1)
3.	$\neg(\Diamond p \to \Diamond q)$	(w)	(1)
4.	$\Diamond p$	(w)	(3)
5.	$\neg\Diamond q$	(w)	(3)
6.	$p \to q$	(v)	(2)
7.	p	(u)	(4)
8.	$\neg q$	(w)	(5)
9.	$\neg q$	(v)	(5)
10.	$\neg q$	(u)	(5)

11.	$\neg p$	(v)	(6)	12.	q	(v)	(6)
					x		

Countermodel: $W = \{w, v, u\}, V(p, v) = 0, V(p, u) = 1, V(q, w) = 0, V(q, v) = 0, V(q, u) = 0$.

Exercise 2.10

(a) Target: $p \to \Box\Diamond p$

1.	$\neg(p \to \Box\Diamond p)$	(w)	(A)
2.	p	(w)	(1)
3.	$\neg\Box\Diamond p$	(w)	(1)
4.	$\neg\Diamond p$	(v)	(3)
5.	$\neg p$	(w)	(4)
	x		

The target sentence is valid.

(b) Target: $\Diamond\Diamond p \to \Diamond p$

1.	$\neg(\Diamond\Diamond p \rightarrow \Diamond p)$	(w)	(A)
2.	$\Diamond\Diamond p$	(w)	(1)
3.	$\neg\Diamond p$	(w)	(1)
4.	$\Diamond p$	(v)	(2)
5.	p	(u)	(4)
6.	$\neg p$	(u)	(3)
	x		

The target sentence is valid.

(c) Target: $\Diamond(p \wedge q) \rightarrow (\Diamond p \wedge \Diamond q)$

1.	$\neg(\Diamond(p \wedge q) \rightarrow (\Diamond p \wedge \Diamond q))$	(w)	(A)
2.	$\Diamond(p \wedge q)$	(w)	(1)
3.	$\neg(\Diamond p \wedge \Diamond q)$	(w)	(1)
4.	$p \wedge q$	(v)	(2)
5.	p	(v)	(4)
6.	q	(v)	(4)

7.	$\neg\Diamond p$	(w)	(3)	8.	$\neg\Diamond q$	(w)	(3)
9.	$\neg p$	(v)	(7)	10.	$\neg q$	(v)	(8)
	x				x		

The target sentence is valid.

(d) Target: $(\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$

1. $\neg((\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q))$ (w) (A)
2. $\Diamond p \wedge \Diamond q$ (w) (1)
3. $\neg\Diamond(p \wedge q)$ (w) (1)
4. $\Diamond p$ (w) (2)
5. $\Diamond q$ (w) (2)
6. p (v) (4)
7. q (u) (5)
8. $\neg(p \wedge q)$ (v) (3)

9. $\neg p$ (v) (8)   10. $\neg q$ (v) (8)
x   11. $\neg(p \wedge q)$ (u) (3)

12. $\neg p$ (u) (11)   13. $\neg q$ (u) (11)
14. $\neg(p \wedge q)$ (w) (3)   x

10. $\neg p$ (w) (14)   10. $\neg q$ (w) (14)

Countermodel: $W = \{w, v, u\}, V(p, w) = 0, V(p, v) = 1, V(p, u) = 0, V(q, v) = 0, V(q, u) = 1$.

(e) Target: $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$

DIY. The sentence is valid.

(f) Target: $\Box\Diamond p \rightarrow \Diamond\Box p$

1.	$\neg(\Box\Diamond p \to \Diamond\Box p)$	(w)	(A)
2.	$\Box\Diamond p$	(w)	(1)
3.	$\neg\Diamond\Box p$	(w)	(1)
4.	$\Diamond p$	(w)	(2)
5.	p	(v)	(4)
6.	$\neg\Box p$	(w)	(3)
7.	$\neg p$	(u)	(6)
8.	$\Diamond p$	(v)	(2)
9.	p	(s)	(8)
10.	$\neg\Box p$	(v)	(3)
11.	$\neg p$	(t)	(10)
	$\vdots$		

This tree grows forever. Manually inspecting the countermodel we could read off from the tree at step 7 shows that it is a genuine countermodel.

Countermodel: $W = \{w, v, u\}, V(p, v) = 1, V(p, u) = 0$.

Exercise 2.11

We could show that the conditional $(A_1 \land \ldots \land A_n) \to B$ is valid. More simply, we can start a tree with individual nodes for the premises and the negated conclusion. If the tree closes, $A_1, \ldots, A_n$ entail B.

A.3. Chapter 3

Exercise 3.1

v has access to no world. So any sentence A is true at *all* (zero) worlds accessible from v.

If this seems strange, remember that $\Box A$ is equivalent to $\neg\Diamond\neg A$. And $\Diamond\neg A$ means that there's an accessible world where $\neg A$ is true. If there are no accessible worlds, then this is false. So $\neg\Diamond\neg A$ is true.

Exercise 3.3

There are infinitely many correct answers for each world. For example: $w_1 : \Diamond\Box p$, $w_2 : \neg p \land \neg q$, $w_3 : \Box p$, $w_4 : \Box q$.

Exercise 3.5

For example: $W = \{w\}$, not wRw, $V(p, w) = 1$.

Exercise 3.6

For example: $\Box(p \lor \neg p) \to (p \lor \neg p)$.

Exercise 3.8

Reflexive yes, serial yes, transitive yes, euclidean no, symmetric no, universal no.

Exercise 3.10

It's true that if R is symmetric and transitive then wRv implies vRw which implies wRw. But this only shows that every world *w that can see some world v* can see itself. What's true is that symmetry, transitivity, *and seriality* together imply reflexivity.

Exercise 3.11

(a) Every world has access only to itself.
(b) No world has access to any world.

A.4. Chapter 4

Exercise 4.3

The claim follows from the Completeness Lemma.

Exercise 4.4

1.	$\Box\neg p \to \neg p$	(T)
2.	$(\Box\neg p \to \neg p) \to (p \to \neg\Box\neg p)$	(Taut)
3.	$p \to \neg\Box\neg p$	(1, 2, MP)
4.	$\Box p \to p$	(T)
5.	$(\Box p \to p) \to ((p \to \neg\Box\neg p) \to (\Box p \to \neg\Box\neg p))$	(Taut)
6.	$(p \to \neg\Box\neg p) \to (\Box p \to \neg\Box\neg p)$	(4, 5, MP)
7.	$\Box p \to \neg\Box\neg p$	(3, 6, MP)

Exercise 4.5

1.	$A \to \Diamond A$	(**T**, Prop. Logic)
2.	$\Diamond A \to \Box\Diamond A$	(**5**)
3.	$\Box\Diamond A \to \Diamond\Box A$	(**M**)
4.	$\Diamond\Box A \to \Box A$	(**5**, Prop. Logic)
5.	$A \to \Box A$	(1, 2, 3, 4, Prop. Logic)

Exercise 4.7

We need to show that everything that's derivable in the axiomatic calculus for S4 is true at every world in every transitive and reflexive Kripke model. From the soundness proof for K, we know that all instances of **A1**–**A3** are true at every world in every Kripke model. From observation 3.3, we know that all instances of **T** are true at every world in every reflexive Kripke model. From observation 3.4, we know that all instances of **4** are true at every world in every transitive Kripke model. So all axioms in the S4-calculus are valid in the class of transitive and reflexive Kripke frames. Since **MP** and **Nec** preserve validity in any class of Kripke frames, it follows that everything that's derivable in the S4-calculus is valid in the class of transitive and reflexive frames.

Exercise 4.8

We have to show that all S4-valid sentences are provable in the axiomatic calculus for S4, which extends the calculus for T by the axiom (schema) $\Box A \to \Box\Box A$. The argument is by contraposition: We suppose that some sentence is not S4-provable and show that it is not S4-valid.

Suppose A is not S4-provable. Then $\{\neg A\}$ is S4-consistent. It follows by Lindenbaum's Lemma that $\{\neg A\}$ is included in some maximal S4-consistent set S. By definition of canonical models, that set is a world in the canonical model M_{S4} for S4. Since $\neg A$ is in S, it follows from the Canonical Model Lemma that $M_{S4}, S \models \neg A$. So $M_{S4}, S \not\models A$.

It remains to show that M_{S4} is reflexive and transitive.

By definition, a world v in a canonical model is accessible from w iff whenever $\Box A \in w$ then $A \in v$. Since the worlds in M_{S4} are maximal S4-consistent sets of sentences, and every such set contains every instance of the **T** schema $\Box A \to A$, there is no world in M_{S4} that contains $\Box A$ but not A. So every world in M_{S4} has access to itself. So M_{S4} is reflexive.

For transitivity, suppose for some worlds w, v, u in M_{S4} we have wRv and vRu. We need to show that wRu. Given how R is defined in M_{S4}, we have to show that u contains all sentences A for which w contains $\Box A$. So let A be an arbitrary sentence for which w contains $\Box A$. Since every world in M_{S4} contains every instance of $\Box A \to \Box\Box A$, we know that w also contains $\Box\Box A$. From wRv, we can infer that v contains $\Box A$. And from vRu, we can infer that u contains A.

Exercise 4.9

The argument closely follows the argument I gave on p.89. The **5** schema would give us $\neg\Box\neg A \to \Box\neg\Box\neg A$. In the minimal logic K, the consequent $\Box\neg\Box\neg A$ entails $\Box(\Box\neg A \to \neg A)$, which entails $\Box\neg A$ by the **GL** schema and **MP**. So if the **5** schema were valid in the logic of provability, then the schema $\neg\Box\neg A \to \Box\neg A$ would also be valid. And that schema is only valid if everything whatsoever is provable, even contradictions.

A.5. Chapter 5

Exercise 5.2

(a) $\mathsf{K}_a(r \lor s)$

r: It is raining; s: It is snowing

(b) $\mathsf{K}_a\, r \lor \mathsf{K}_a\, s$
r: It is raining; s: It is snowing

(c) $\mathsf{K}_b\, r \lor \mathsf{K}_b\, \neg r$
r: It is raining

(d) $\mathsf{K}_c \neg \mathsf{K}_c\, r$
r: It is raining

(e) $\mathsf{K}_a(\mathsf{K}_b\, r \lor \mathsf{K}_b\, \neg r)$
r: It is raining

Exercise 5.3

One possible answer: There are many propositions that I don't know, and that don't logically follow from things I know. And for many of these propositions I *know* that that they don't follow from things I know. (For example, I know that it doesn't follow from anything I know that Edinburgh is in Italy.) Let p be some such proposition. Since I know that $\neg\, \mathsf{K}^+\, p$ is true, I can easily figure out that $\mathsf{K}^+\, p \to p$ is true as well (by the truth-table for the arrow). So $\mathsf{K}^*(\mathsf{K}^+\, p \to p)$. If $\mathsf{K}^*(\mathsf{K}^+\, A \to A) \to \mathsf{K}^+\, A$ were valid, it would follow that p does after all follow from what I know!

Exercise 5.4

see https://plato.stanford.edu/entries/dynamic-epistemic/appendix-B-solutions.html

Exercise 5.7

You can find the trees for (a)–(d) on https://www.wolfgangschwarz.net/trees/ if you write K as a box and M as a diamond.

Exercise 5.10

(a) and (b) are valid, (c) and (d) are invalid. One way of testing this is with the method of trees; but you'll need to keep track of both accessibility relations. To illustrate, here is a tree proof for (a).

1.	$\neg(\mathsf{M}_1\,\mathsf{K}_2\,A \to \mathsf{M}_1\,A)$	(w)	(Ass.)
2.	$\mathsf{M}_1\,\mathsf{K}_2\,A$	(w)	(1)
3.	$\neg\,\mathsf{M}_1\,A$	(w)	(1)
4.	wR_1v		(2)
5.	$\mathsf{K}_2\,A$	(v)	(2)
6.	$\neg A$	(v)	(3,4)
7.	vR_2v		(Refl.)
8.	A	(v)	(5,7)
	x		

The tree for (c) doesn't close:

1.	$\neg(\mathsf{M}_1\,\mathsf{K}_2\,A \to \mathsf{M}_2\,\mathsf{K}_1\,A)$	(w)	(Ass.)
2.	$\mathsf{M}_1\,\mathsf{K}_2\,A$	(w)	(1)
3.	$\neg\,\mathsf{M}_2\,\mathsf{K}_1\,A$	(w)	(1)
4.	wR_1v		(2)
5.	$\mathsf{K}_2\,A$	(v)	(2)
6.	vR_2v		(Refl.)
7.	A	(v)	(5,6)
8.	wR_2w		(Refl.)
9.	$\neg\,\mathsf{K}_1\,A$	(w)	(3,8)
10.	wR_1u		(9)
11.	$\neg A$	(u)	(9)

We could add a few more applications of Reflexivity, but the tree would remain open. It also gives us a countermodel: let $W = \{w, v, u\}$; w has 1-access to v and u; each world has 1- and 2-access to itself; $V(p, v) = 1$ and $V(p, u) = 0$. In this model, at world w, $\mathsf{M}_1\,\mathsf{K}_2\,p$ is true while $\mathsf{M}_2\,\mathsf{K}_1\,p$ is false.

Cases (b) and (d) are similar.

Exercise 5.11

The **5**-schema for E_G states that $\neg \mathsf{E}_G \neg A \to \mathsf{E}_G \neg \mathsf{E}_G \neg A$. To show that this is invalid, we need to find a case where some instance of $\neg \mathsf{E}_G \neg A$ is true while $\mathsf{E}_G \neg \mathsf{E}_G \neg A$ is false. Let's take the simplest instance, with $A = p$, and let's assume the relevant group has two agents. Consider a world w at which $\mathsf{K}_1 \neg p$ and $\neg \mathsf{K}_2 \neg p$ are true. By the assumption that **5** is valid for K_i, $\mathsf{K}_2 \neg \mathsf{K}_2 \neg p$ is also true at w. But $\mathsf{K}_1 \neg \mathsf{K}_2 \neg p$ can be false (at w). If it is, then $\neg \mathsf{E}_G \neg p$ is true at w while $\mathsf{E}_G \neg \mathsf{E}_G \neg p$ is false.

Exercise 5.12

No, a transitive, serial, and euclidean relation is not always symmetric. (Counterexample: wRv, vRv.) This means that **B** (which corresponds to symmetry) is not valid in KD45.

Exercise 5.13

You can e.g. do a tree proof, using B as the box.

Exercise 5.14

By **PI**, $\mathsf{B}\,A \to \mathsf{K}\,\mathsf{B}\,A$ is valid. By **KB**, so is $\mathsf{K}\,\mathsf{B}\,A \to \mathsf{B}\,\mathsf{B}\,A$. By propositional logic, it follows that $\mathsf{B}\,A \to \mathsf{B}\,\mathsf{B}\,A$ is valid, which is principle **4** for belief.

By **NI**, $\neg \mathsf{B} \neg A \to \mathsf{K} \neg \mathsf{B} \neg A$ is valid. By **KB**, so is $\mathsf{K} \neg \mathsf{B} \neg A \to \mathsf{B} \neg \mathsf{B} \neg A$. By propositional logic, it follows that $\neg \mathsf{B} \neg A \to \mathsf{B} \neg \mathsf{B} \neg A$ is valid, which is principle **5** for belief.

A.6. Chapter 6

Exercise 6.1

For example: $\mathsf{O}\,\neg$, or $\neg\,\mathsf{P}$.

Exercise 6.3

(a): $\Box(\mathsf{OK} \to (\Box(\mathsf{OK} \to A) \to A))$.

Exercise 6.4

$\mathsf{P}\,A$ could be defined as $\neg\Box(\mathsf{OK} - > \neg A)$, or more simply (and equivalently) as

$\Diamond(\mathsf{OK} \land A)$.

Exercise 6.5

On the absolutist conception, wRv means that v is ideal. So transitivity (if wRv and vRu then wRu) and euclidity (if wRv and wRu then vRu) both state that if v is ideal and u is ideal then u is ideal.

Exercise 6.6

Consider the example from the text, where w is the actual world (in the UK) and u is a w-accessible world at which everyone drives on the left although the law says that one must drive on the right. A typical world accessible from u will be a world at which people drive on the right. This world will not be accessible from w. So we have a counterexample to transitivity. We also have a counterexample to euclidity because we have wRu and wRu but not uRu. (Euclidity entails shift reflexivity.)

Exercise 6.8

In the described situation, it ought to be the case that Amy is either obligated to help Betty or obligated to help Carla, but Amy is neither obligated to help Betty nor to help Carla. So if p translates 'Amy helps Betty' and q 'Amy helps Carla', we seem to have $\mathsf{O}(\mathsf{O}\, p \lor \mathsf{O}\, q)$ and $\neg \mathsf{O}\, p$ and $\neg \mathsf{O}\, q$. But these assumptions are inconsistent in KD5. You can draw a KD5-tree (using the Seriality and Euclidity rules) starting with $\mathsf{O}(\mathsf{O}\, p \lor \mathsf{O}\, q)$ and $\neg \mathsf{O}\, p$ and $\neg \mathsf{O}\, q$ on which all branches close. This shows that there is no world in any euclidean and serial model at which the three assumptions are true.

Exercise 6.9

(a) I argue by contraposition. Suppose $\mathsf{O}\,\mathsf{O}\, A \to \mathsf{O}\, A$ is invalid on a frame. This means that at some world w in some model M based on the frame, (some instance of) $\mathsf{O}\,\mathsf{O}\, A$ is true while $\mathsf{O}\, A$ is false. It follows that there is a world accessible from w at which A is false and $\mathsf{O}\, A$ true. So $\mathsf{O}\, A \to A$ is false at v. So $\mathsf{O}(\mathsf{O}\, A \to A)$ is false at w. (You could also give a tree proof with the K-rules showing that **U** entails **C4**.)

(b) This direction is hard. It is not enough to give a model in which some instance of **C4** is true at some world while the corresponding instance of **U** is false. For a counterexample, you need to give a *frame* on which every instance of **C4** is valid but not every instance of **U**. Here is one such frame: W is the set of real numbers; wRv holds iff $w < v$.

Exercise 6.10

Use https://www.wolfgangschwarz.net/trees/. (Write O as a box and P as a diamond, and make the accessibility relation serial.)

Exercise 6.14

Simply replace 'all' in the semantics for $\mathsf{O}(B/A)$ with 'some'.

Exercise 6.15

Ross's Paradox: 'Alice must be in the office or in the library' seems to imply that Alice might be in the office and that she might be in the library.

The Paradox of Free Choice: 'Alice might be in the office or in the library' seems to imply that Alice might be in the office and that she might be in the library.

Exercise 6.16

Whenever $X \in N(w)$ then all sets that have X as a subset are in $N(w)$.

Exercise 6.17

For every world w, every member of $N(w)$ contains w.

Exercise 6.18

In Kripke semantics, $\Box p$ and $\Box q$ together entail $\Box(p \land q)$. But if the probability of p is above the threshold and the probability of q is above the threshold, it does not follow that the probability of $p \land q$ is above the threshold. For example, we could have $\text{Pr}(p) = 0.95$, $\text{Pr}(q) = 0.94$, and $\text{Pr}(p \land q) = 0.95 \times 0.94 = 0.893$.

A.7. Chapter 7

Exercise 7.1

(a) $\mathsf{H}\neg p$
p: It is warm

(b) $\mathsf{F}\,p$
p: There is a sea battle

(c) $\mathsf{F}\neg\mathsf{P}\,p$ (or, perhaps, $\neg\mathsf{F}\,\mathsf{P}\,p$)
p: There is a sea battle

(d) $\mathsf{F}(p \lor \mathsf{P}\,q)$
p: It is warm

(e) $\neg\mathsf{P}\,p \to \neg\mathsf{F}\,q$
p: You study, q: you pass the exam

(f) $\mathsf{P}(p \land q)$
p: I am having tea, q: the door bell rings

Exercise 7.2

(a), (c), (f), (g), and (h) are true, (b), (d), and (e) are false.

Exercise 7.5

Since G is the backward-looking box, the **4** schema for G corresponds to transitivity of the 'later than' relation $\succ$: if $t \succ s$ and $s \succ r$ then $t \succ r$. We can rewrite '$t \succ s$' as '$s \prec t$'. Similarly, '$s \succ r$' is equivalent to '$r \prec s$' and '$t \succ r$' to '$r \prec t$'. So transitivity of $\succ$ requires that if $s \prec t$ and $r \prec s$, then $r \prec t$. This evidently follows from transitivity of $\prec$.

Exercise 7.7

If time is transitive and circular, then it is neither asymmetric nor irreflexive.

Exercise 7.10

(a) For example, $\mathsf{G}\,A \to \mathsf{F}\,A$.
(b) For example, $\mathsf{H}\,A \to \mathsf{P}\,A$.

(c) No schema corresponds to the class of frames with a last time. If we also assume transitivity and weak connectedness (see page 145), then $\mathsf{G}(A \land \neg A) \lor \mathsf{F}\,\mathsf{G}(A \land \neg A)$ works.

(d) No schema corresponds to the class of frames with a first time. If we also assume transitivity and weak connectedness, then $\mathsf{H}(A \land \neg A) \lor \mathsf{P}\,\mathsf{H}(A \land \neg A)$ works.

Exercise 7.11

We have to show that $\mathsf{F}\,A \to \mathsf{F}\,\mathsf{F}\,A$ is valid on all and only the dense frames. A frame is dense if whenever $t < s$ then there is a point r such that $t < r$ and $r < s$.

So assume a frame is dense and suppose for reductio that some instance of $\mathsf{F}\,A \to \mathsf{F}\,\mathsf{F}\,A$ is false at some point t in some model M based on that frame. Then $\mathsf{F}\,A$ is true at t and $\mathsf{F}\,\mathsf{F}\,A$ is false. Since $\mathsf{F}\,A$ is true at t, it follows by definition 7.2 that A is true at some point s such that $t < s$. By density, there is a point r such that $t < r < s$. But since A is true at s, $\mathsf{F}\,A$ is true at r, and so $\mathsf{F}\,\mathsf{F}\,A$ is true at t; contradiction.

In the other direction, we have to show that if a frame isn't dense then some instance of $\mathsf{F}\,A \to \mathsf{F}\,\mathsf{F}\,A$ is false at some point t in some model M based on that frame. We take the simplest instance $\mathsf{F}\,p \to \mathsf{F}\,\mathsf{F}\,p$. If a frame isn't dense then there are points t, s such that $t < s$ and no point lies in between t and s. Let V be an interpretation function that makes p true at s and false everywhere else. Then $\mathsf{F}\,p$ is true at t but $\mathsf{F}\,\mathsf{F}\,p$ is false. So $\mathsf{F}\,p \to \mathsf{F}\,\mathsf{F}\,p$ is false at t.

Exercise 7.12

Without assumptions about the flow of time there is no way to express in $\mathfrak{L}_T$ that p is true at all times (or at some time). In linear flows, $p \land \mathsf{H}\,p \land tGp$ does the job.

Exercise 7.14

(a)–(d) are valid, (e) is invalid.

To show that a schema is valid, assume for reductio that there is some time t on some history H in some model M at which the schema is false. Then (repeatedly) use definition 7.3 to derive a contradiction.

For (e), consider a model with three times t, s, r such that $s \prec t$, $r \prec t$, and neither

$s \prec r$ nor $r \prec s$. Let q be true at s and false at the other two times. $\mathsf{P}\,q \to \Box\,\mathsf{P}\,\Diamond q$ is false at t on the history $\langle s, t \rangle$.

Exercise 7.16

Ockham-entailment is stronger than super-entailment: whenever A Ockham-entails B, then A super-entails B, but not the other way around.

Suppose A Ockham-entails B. Let t be any time in any temporal model at which A is true, i.e.: true relative to all histories through t. Since A Ockham-entails B, B is true at t relative to all histories through t. So A super-entails B.

But suppose A super-entails B. Let t be any time on any history h in any temporal model at which A is true. We can't infer that B is true at t on h, for A may be false at t relative to other histories h'. So we can't infer that A Ockham-entails B. Indeed, $\mathsf{F}\,p$ super-entails $\Box\,\mathsf{F}\,p$, but $\mathsf{F}\,p$ does not Ockham-entail $\Box\,\mathsf{F}\,p$.

Exercise 7.18

$\mathsf{N}\,A \to A$ is valid. So $\Box(\mathsf{N}\,A \to A)$. By assumption (a), we could infer $\mathsf{G}\,\Box(\mathsf{N}\,A \to A)$. Assumption (b) would tell us that $\mathsf{G}(\Box(\mathsf{N}\,A \to A) \to (\mathsf{N}\,A \to A))$. By the **FK** principle and *modus ponens*, this implies $\mathsf{G}\,\Box(\mathsf{N}\,A \to A) \to \mathsf{G}(\mathsf{N}\,A \to A)$. So if we make both assumptions, we can infer $\mathsf{G}(\mathsf{N}\,A \to A)$, which is equivalent to the claim that whatever is now true is always going to be true.

It is not obvious which of the two assumptions should be rejected.

A.8. Chapter 8

Exercise 8.1

For example: $\neg A \,\exists\, A$ or $(B \vee \neg B) \,\exists\, A$.

Exercise 8.2

$W = \{w\}$, $R = \emptyset$, $V(p, w) = 1$, $V(q, w) = 0$.

Exercise 8.5

All the examples from section 1 also work in the subjunctive mood. For **P4** and **P5** this yields:

- If our opponents had been cheating, we would never have found out. Therefore: If we had found out that our opponents are cheating, then they wouldn't have been cheating.
- If you had added sugar to your coffee, it would have tasted good. Therefore: If you had added sugar and vinegar to your coffee, it would have tasted good.

Exercise 8.6

Suppose $A \to B$ is assertable. Then $A \to B$ is known. So $\mathsf{K}(A \to B)$. In S4, it follows that $\mathsf{K}\,\mathsf{K}(A \to B)$. So the epistemically strict conditional $\mathsf{K}(A \to B)$ is assertable. Conversely, if $\mathsf{K}(A \to B)$ is assertable, then it is known; so $\mathsf{K}\,\mathsf{K}(A \to B)$. In S4, it follows that $\mathsf{K}(A \to B)$. So $A \to B$ is assertable.

Exercise 8.8

Suppose $A \mathrel{\Box\!\!\rightarrow} B$ is true at some world w in some model M. So B is true at all the closest A-worlds to w. Now either A is true at w or A is false at w. If A is false at w, then $A \to B$ is true at w. If A is true at w, then w is one of the closest A-worlds to w, by Weak Centring; so B is true at w; and so $A \to B$ is true at w. Either way, then, $A \to B$ is true at w.

Exercise 8.9

If A is true at no worlds, then $Min^{<_w}(\{u : M, u \models A\})$ is the empty set. So it is vacuously true that $M, v \models B$ for all $v \in Min^{<_w}(\{u : M, u \models A\})$.

Exercise 8.13

'All dogs are barking': $\forall x(Dx \to Bx)$
'Some dogs are barking': $\exists x(Dx \land Bx)$
'Most dogs are barking' cannot be translated in terms of $\mathsf{M}\,x$. We need a binary quantifier: $\mathsf{M}\,x(Bx/Dx)$

A.9. Chapter 9

Exercise 9.1

(a) $Srj \wedge Szj$; r: Keren, z: Keziah, j: Jemima, S: – is a sister of –
(b) $\forall x(Mx \rightarrow Ox)$; M: – is a myriapod, O: – is oviparous
(c) $\exists xDx$; D: – is at the door
(d) $\neg\forall x(Sx \rightarrow Lxl)$; l: logic; S: – is a student, L: – likes –
(e) $\forall x(Sx \wedge Lxl \rightarrow \exists yLxy)$; l: logic; S: – is a student, L: – likes –

Exercise 9.3

For both cases, use Fx as sentence A, and $\neg Fx$ as B.

Exercise 9.4

Let M_1 be a model with two worlds; each world can see the other but not itself; at both worlds, all sentence letters are false. Let M_2 be a model with a single world that can see itself and at which all sentence letters are false. The very same $\mathfrak{L}_M$-sentences are true at all worlds in these models (as a simple proof by induction shows). But the first model is irreflexive and the second isn't.

Exercise 9.5

For example: $\forall wRww$ states that R is reflexive; as the answer to the previous exercise shows, this can't be expressed in $\mathfrak{L}_M$. There are many other examples.

Exercise 9.7

If a sentence is valid in first-order predicate logic, then a fully expanded tree for the sentence will close and show that the sentence is valid. But if a sentence is not valid, the tree might grow forever. There is no algorithm for detecting whether a tree will grow forever. So there are trees that grow forever but you can't tell that they will grow forever.

Exercise 9.8

(a) $\Box Fa$
a: John, F: – is hungry.

(b) $\Box\forall x(Fx \to Gx)$
F: – is a cyclist, G: – has legs.

Also correct (but different in meaning): $\forall x(Fx \to \Box Gx)$.
Close but incorrect: $\forall x\Box(Fx \to Gx)$.

(c) $\forall x(Fx \to \Diamond Gx)$
F: – is a day, G: – is our last day.

The English sentence could also be understood as $\Diamond\forall x(Fx \to Gx)$, but that would be a very strange thing to say.

(d) $a = b \to \Diamond\forall x(Fx \to Hxb)$
a: water, b: H_2O, F: – is a cucumber, H: – contains –.

It's not obvious that 'water' and 'H_2O' function as names. So you might also try to translate them as predicates:

$\Box\forall x(Gx \leftrightarrow Jx) \to \Diamond\forall x(Fx \to \exists y(Jy \land Hxy))$
F: – is a cucumber, H: – contains –, G: – is water, J: – is H_2O.

Here $\Box(Gx \leftrightarrow Jx)$ is an attempt to say that water is H_2O.

(e) $\forall x\, \mathsf{O}(Fx \to Gx)$
F: – wants to leave early, G: – leaves quietly.

Even better, if we can use the conditional obligation operator: $\forall x\, \mathsf{O}(Gx/Fx)$

These aren't too far off either: $\forall x(Fx \to\ \mathsf{O}\, Gx)$, $\mathsf{O}\,\forall x(Fx \to Gx)$

(f) $\forall x(\exists y(Fy \land Hxy) \to\ \mathsf{P}\, Gx)$
F: – is a ticket, G: – enters, H: – bought –.

Perhaps even better: $\forall x\, \mathsf{P}(Gx/\exists y(Fy \land Hxy))$.

You could translate 'bought a ticket' as a simple predicate here; you could also use a temporal operator to account for the past tense of 'bought' (but it's confusing to use two different kinds of 'P' in one sentence).

(g) $\mathsf{F}\,\forall x(\mathsf{N}\, Fx \to Gx)$
F: – is rich, G: – is poor.

The non-trivial reading of the sentence can't be translated without the 'now' operator N.

Exercise 9.11

1. $b = c$
2. A
3. $b = b$ (Id.)
4. $c = b$ (1, 3, LL (first version))
5. $A[b//c]$ (4, 2, LL (first version))

Exercise 9.12

(a) $\exists x \exists y(Fx \land Fy \land x \neq y \land \forall z(Fz \to (z = x \lor z = y)))$

(b) $\forall x \forall y \forall z \forall v(Fx \land Fy \land Fz \land Fv \to (x = y \lor x = z \lor x = v \lor y = z \lor y = v \lor z = v))$

Exercise 9.13

The *de dicto* reading of (a) can be translated as

$$\Diamond \exists x(Px \land \forall y(Py \to x = y) \land x = c),$$

where 'P' translates '– is 45th US President' and 'c' denotes Hilary Clinton. The *de re* reading can be translated as

$$\exists x(Px \land \forall y(Py \to x = y) \Diamond x = c).$$

A.10. Chapter 10

Exercise 10.1

(b), (c), (d), and (e) are true; (a), (f), (g) are false.

Exercise 10.3

(a) $W = \{w\}$, wRw, $D = \{\text{Alice}\}$, $V(F, w) = \{\text{Alice}\}$, $V(G, w) = \emptyset$

(b) $W = \{w, v\}$, wRw and wRv, $D = \{\text{Alice, Bob}\}$, $V(F, w) = \{\text{Alice}\}$, $V(F, v) = \{\text{Bob}\}$

(c) $W = \{w, v\}$, wRw and wRv, $D = \{\text{Alice, Bob}\}$, $V(F, w) = \{\text{Alice}\}$, $V(F, v) = \emptyset$

(d) $W = \{w, v\}$, wRw and wRv, $D = \{\text{Alice, Bob}\}$, $V(P, w) = \{\text{Alice}\}$, $V(P, v) = \emptyset$, $V(Q, w) = \emptyset$, $V(Q, v) = \emptyset$

Exercise 10.4

$\Box\forall x\exists y(x = y) \to \forall x\Box\exists y(x = y)$ is an instance of the Converse Barcan Formula. If we read the box as a relevant kind of circumstantial necessity, and Loafy could have failed to exist, the consequent of this conditional is false. But the antecedent is true.

Exercise 10.5

(1) is equivalent to the Barcan Formula, (4) to the Converse Barcan Formula. (2) is highly implausible. (1) and (4) are often regarded as implausible, for the reasons I discuss in the text. (3) is as plausible or implausible as the Converse Barcan Formula. (Consider the instance $\exists x\Box\neg\exists y(x = y) \to \Box\exists x\neg\exists y(x = y)$.

Exercise 10.9

We would assume that (i) the name g picks out a statue at all accessible worlds, (ii) l picks out a lump of clay at all accessible worlds, and (iii) at the actual world, l and g pick out the same thing: the statue-shaped lump on the shelf.

Exercise 10.10

To render $\forall x\forall y(x = y \to \Box x = y)$ valid, we can stipulate that for any variables x, y, and worlds w, v such that wRv, and any assignment function g, if $g(x, w) = g(y, w)$ then $g(x, v) = g(y, v)$. (We should make an analogous stipulation for the interpretation function V with respect to names.)

Exercise 10.11

Translation: $\exists x(Tx \land Wx \land \neg KWx \land \neg K\neg Wx)$, where T translates '– is a ticket' and '– will win'.

If variables are directly referential, then this sentence is true in any scenario in which I don't know which ticket will win.

In counterpart semantics, we could assume that the sentence makes *being the winning ticket* a salient respect of similarity, so that a ticket t at an epistemically accessible world w is a counterpart of the winning ticket at the actual world iff t is winning at w. This renders the sentence false in any situation in which I know that there is a winning ticket (as the first part of the sentence conveys).