Logic 2: Modal Logic

Lecture 2

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Review

Review

- We've added the box $\Box$ and the diamond $\Diamond$ to the language of propositional logic.
- The box often represents some kind of necessity, the diamond some kind of possibility.
- We've talked about how to translate from English into the language of modal propositional logic.

Validity and logical validity

Validity and logical validity

What do we mean when we say that an argument is **valid**?

> An argument is valid if there is no conceivable scenario in which the premises are true and the conclusion is false.

Check your understanding:

- Can an argument be valid if its conclusion is false?
- Is this argument valid? 'It is cold and it is not cold. Therefore: It is raining.'

Validity and logical validity

It is hot and windy.	It is hot.
It is hot.	It is not cold.

Only the first argument is **logically valid**.

An argument is logically valid if its validity does not turn on the meaning of the non-logical expressions.

- Re-interpret ‘hot’ to mean *cloudy*.
- The first argument remains valid, the second becomes invalid.

Validity and logical validity

An argument is valid if there is no conceivable scenario in which the premises are true and the conclusion false.

> An argument is logically valid if there is no conceivable scenario in which the premises are true and the conclusion is false, under any (re-)interpretation of the non-logical expressions.

Validity and logical validity

In modal logic, we treat the box and the diamond as logical expressions.

It is possible that it is raining.	$\Diamond r$
It is certain that we will get wet if it is raining.	$\Box(r \rightarrow w)$
It is possible that we will get wet.	$\Diamond w$

There is no conceivable scenario in which the premises are true and the conclusion is false, under any interpretation of the non-logical expressions.

The turnstile

The turnstile

We say that some sentences $A_1, A_2, \ldots$ **(logically) entail** a sentence B if the argument from $A_1, A_2, \ldots$ to B is logically valid.

Let's introduce an abbreviation.

$$A_1, A_2, \ldots \models B \iff A_1, A_2, \ldots \text{ logically entail } B.$$

$$\iff \text{There is no conceivable scenario in which } A_1, A_2, \ldots \text{ are all true while } B \text{ is false, on any interpretation of the non-logical expressions.}$$

The turnstile

- It is hot and windy $\models$ It is hot.
- $p \land q \models p$.
- $\Diamond p, \Box(p \to q) \models \Diamond q$.

$A_1, A_2 \ldots \models B$ iff there is no conceivable scenario and interpretation of non-logical expressions that would make all of $A_1, A_2, \ldots$ true and B false.

Informally, the turnstile says “you can’t make everything on the left true while making everything on the right false”.

A special case: $\models B$. This says that B is (logically) valid.

A sentence is (logically) valid iff there is no conceivable scenario in which it is false, on any interpretation of the non-logical expressions.

The turnstile

- $\models p \rightarrow p$
- $\models \forall x Fx \vee \exists x \neg Fx$
- $\models \Box(p \wedge q) \rightarrow \Box p$

Proofs

Proofs

$$\begin{array}{l} \Diamond r \\ \Box(r \rightarrow w) \\ \hline \Diamond w \end{array}$$

There is no conceivable scenario in which the premises are true and the conclusion is false, under any interpretation of the non-logical expressions.

This is a **model-theoretic** assessment of the argument.

A **proof-theoretic** assessment would check if the conclusion is derivable from the premises by certain rules of inference.

Proofs

There are many ways to do formal proofs.

1	SHØW: $1 : \Box\varphi \to \Box\psi$	$[3, LCOND]$
2	$1 : \Box\varphi$	$ass.$
3	SHØW: $1 : \Box\psi$	$[k+1, LRED]$
4	$1 : \neg\Box\psi$	$ass.$
5	SHØW: $1.1 : \varphi \wedge \neg\psi$	$[i+1, LE_2]$
6	$1.1 : \neg\varphi \wedge \psi$	$ass.$
7	$1.1 : \neg\varphi$	$(6, L\alpha E)$
8	$1.1 : \psi$	$(6, L\alpha E)$
9	SHØW: $1.1 : \varphi \leftrightarrow \psi$	
	$\mathcal{D}[\sigma/1.\sigma]$	
i	$1.1 : \varphi$	$(8, 9)$
$i+1$	$\bot$	$(7, i, L\bot I)$
$i+2$	$1.1 : \varphi$	$(5, L\alpha E)$
$i+3$	$1.1 : \neg\psi$	$(5, L\alpha E)$
$i+4$	SHØW: $1.1 : \varphi \leftrightarrow \psi$	
	$\mathcal{D}[\sigma/1.\sigma]$	
k	$1.1 : \psi$	$(i+2, i+4)$
$k+1$	$\bot$	$(i+3, k, L\bot I)$

Proofs

There are many ways to do formal proofs.

1	$\quad p \to q$	ass.
2	$\quad\quad q \to r$	ass.
3	$\quad\quad\quad p$	ass.
4	$\quad\quad\quad p \to q$	1, (rep.)
5	$\quad\quad\quad q$	3, 4, $(\to E)$
6	$\quad\quad\quad q \to r$	2, (rep.)
7	$\quad\quad\quad r$	5, 6, $(\to E)$
8	$\quad\quad p \to r$	3–7 $(\to I)$
9	$\quad (q \to r) \to (p \to r)$	2–8, $(\to I)$
10	$(p \to q) \to ((q \to r) \to (p \to r))$	1–9, $(\to I)$

Proofs

There are many ways to do formal proofs.

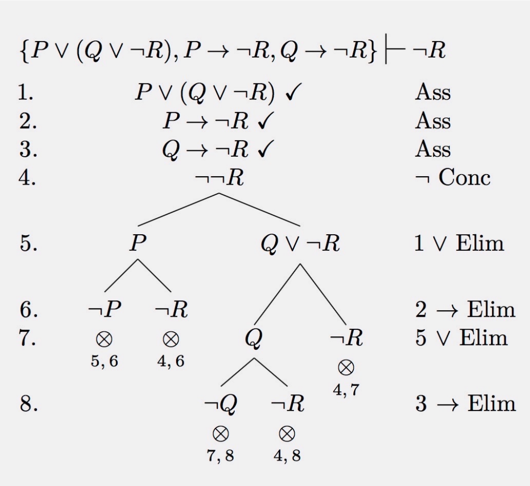
$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \to (B \to C)}\,{}^{2} \qquad \dfrac{\dfrac{}{A \land B}\,{}^{1}}{A}}{B \to C} \qquad \dfrac{\dfrac{}{A \land B}\,{}^{1}}{B}}{\dfrac{C}{A \land B \to C}\,{}^{1}}}{(A \to (B \to C)) \to (A \land B \to C)}\,{}^{2}
$$

Proofs

There are many ways to do formal proofs.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(I) \quad \dfrac{}{C \vdash C}\,(I)}{B \lor C \vdash B, C}\,(\lor L)}{B \lor C \vdash C, B}\,(PR)}{B \lor C, \neg C \vdash B}\,(\neg L) \quad \dfrac{}{\neg A \vdash \neg A}\,(I)}{(B \lor C), \neg C, (B \to \neg A) \vdash \neg A}\,(\to L)}{(B \lor C), \neg C, ((B \to \neg A) \land \neg C) \vdash \neg A}\,(\land L_1)}{(B \lor C), ((B \to \neg A) \land \neg C), \neg C \vdash \neg A}\,(PL)}{(B \lor C), ((B \to \neg A) \land \neg C), ((B \to \neg A) \land \neg C) \vdash \neg A}\,(\land L_2)}{(B \lor C), ((B \to \neg A) \land \neg C) \vdash \neg A}\,(CL)}{((B \to \neg A) \land \neg C), (B \lor C) \vdash \neg A}\,(PL) \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,(I)}{\vdash \neg A, A}\,(\neg R)}{\vdash A, \neg A}\,(PR)}{((B \to \neg A) \land \neg C), (A \to (B \lor C)) \vdash \neg A, \neg A}\,(\to L)$$

There are many ways to do formal proofs.



Proofs

There are many ways to do formal proofs.

1.	$((P \rightarrow ((P \rightarrow P) \rightarrow P)) \rightarrow ((P \rightarrow (P \rightarrow P)) \rightarrow (P \rightarrow P)))$	by Ax2
2.	$(P \rightarrow ((P \rightarrow P) \rightarrow P))$	by Ax1
3.	$((P \rightarrow (P \rightarrow P)) \rightarrow (P \rightarrow P))$	from 2, 1 by MP
4.	$(P \rightarrow (P \rightarrow P))$	Ax1
5.	$(P \rightarrow P)$	from 4, 3 by MP

Proofs

In the early days of formal logic, the only known method of formal proof was the **axiomatic method**.

In the axiomatic method, you lay down some axioms and inference rules.

A proof is a list of sentences each of which is either an axiom or follows from earlier sentences by one of the rules.

Proofs

The Frege-Łukasiewicz axiomatization of propositional logic:

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**MP**) From A and $A \to B$ one may infer B

All truth-functional tautologies are derivable from A1–A3 by MP.

Proofs

Axiomatic proofs are often hard to find.

1.	$p \to ((p \to p) \to p)$	(A1)
2.	$(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$	(A2)
3.	$(p \to (p \to p)) \to (p \to p)$	(1, 2, MP)
4.	$p \to (p \to p)$	(A1)
5.	$p \to p$	(3, 4, MP)

Proofs

In the 1920s and 1930s, C.I. Lewis put forward a range of axiomatic “systems” of modal propositional logic, which he called S1–S5.

Each of his “systems” consisted of some axioms and rules.

In 1933, Kurt Gödel gave the following axiomatization of S4:

(**PL**) The axioms of propositional logic

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**T**) $\Box A \to A$

(**4**) $\Box A \to \Box\Box A$

(**MP**) From A and $A \to B$ one may infer B

(**Nec**) From A one may infer $\Box A$

Proofs

We will want to make sure that the proof-theoretic and the model-theoretic assessment of arguments agree.

A sentence should be provable iff it is true in all conceivable scenarios under all interpretations of the non-logical vocabulary.