Logic 2: Modal Logic

Lecture 1

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Course info

Course info

- This is an intermediate logic course focusing on modal logic.
- You must have taken Logic 1 or something equivalent.
- Logic 1 was easy. This course will be harder.

Course info

Website, Readings, Exercises

- The website for this course is `wolfgangschwarz.net/logic2`.
- Each week I will post extensive lecture notes with exercises.
- You are strongly advised to read the notes and attempt to answer the exercises.
- Work in groups!

Course info

Tutorials

- Tutorials run for two hours.
- In the first hour, we go through the answers to selected exercises.
- The second hour is an open Q&A session, often with more answers to exercises.
- Only the first hour is compulsory.

Course info

Assessment

- 20% First take-home test (7-10 October)
- 30% Second take-home test (4-7 November)
- 50% Final exam (sometime in December)

What is modal logic?

What is modal logic?

In propositional logic, we can formalize this argument:

It is either raining or snowing.	$r \vee s$
It is not snowing.	$\neg s$
It is raining.	$r.$

What is modal logic?

But we can't adequately formalize this:

All birds have feathers.	p
All penguins are birds.	q
All penguins have feathers.	r

We need to **extend our formal language**.

What is modal logic?

Enter predicate logic.

All birds have feathers.	$\forall x(Bx \to Fx)$
All penguins are birds.	$\forall x(Px \to Bx)$
All penguins have feathers.	$\forall x(Px \to Fx)$

What is modal logic?

We still can't adequately formalize other arguments.

It is possible that it is raining.	p
It is certain that we will get wet if it is raining.	q
It is possible that we will get wet.	r

We need to **extend our formal language**.

What is modal logic?

Enter modal logic.

It is possible that it is raining.	$\Diamond r$
It is certain that we will get wet if it is raining.	$\Box(r \to w)$
It is possible that we will get wet.	$\Diamond w$

What is modal logic?

In basic modal logic, we add two new operators to our formal language:

- the box $\Box$
- the diamond $\Diamond$

If we add $\Box$ and $\Diamond$ to the standard language of propositional logic, we get modal propositional logic.

If we add $\Box$ and $\Diamond$ to the standard language of predicate logic, we get modal predicate logic.

What is modal logic?

The box and the diamond **mean different things in different applications of modal logic.**

The box can mean

- It is certain that ...
- It is known that ...
- Alice knows that ...
- It is inevitable that ...
- It could not have failed to be the case that ...
- It is always going to be the case that ...
- It is obligatory that ...
- It is mathematically provable that ...
- ...

What is modal logic?

Branches of modal logic

- Epistemic logic (reasoning about certainty, knowledge, belief)
- Deontic logic (reasoning about permission and obligation)
- Alethic logic (reasoning about what could have been the case)
- Temporal logic (reasoning about the flow of time)
- Provability logic (reasoning about mathematical provability)
- ...

Entailment and validity

Entailment and validity

An argument is valid if there is no conceivable scenario in which the premises are true and the conclusion false.

An argument is logically valid if it is valid in virtue of the meaning of the logical expressions.

> An argument is logically valid if there is no conceivable scenario in which the premises are true and the conclusion is false, under any (re-)interpretation of the non-logical expressions.

Entailment and validity

Natural language	Logic
All myriapods are oviparous.	$\forall x(Mx \rightarrow Ox)$
Some arthropods are myriapods.	$\exists x(Ax \rightarrow Mx)$
Some arthropods are oviparous.	$\exists x(Ax \rightarrow Ox)$

There is no conceivable scenario in which the premises are true and the conclusion is false, under any interpretation of the non-logical expressions.

Entailment and validity

$\forall x(Mx \rightarrow Ox)$
$\exists x(Ax \rightarrow Mx)$

$\exists x(Ax \rightarrow Ox)$

Because our topic is logical validity, we mostly work with formal languages in which the non-logical expressions don't have a fixed meaning.

Entailment and validity

In modal logic, we treat the box and the diamond as logical expressions.

It is possible that it is raining.	$\Diamond r$
It is certain that we will get wet if it is raining.	$\Box(r \rightarrow w)$
It is possible that we will get wet.	$\Diamond w$

There is no conceivable scenario in which the premises are true and the conclusion is false, under any interpretation of the non-logical expressions.

The double-barred turnstile

The double-barred turnstile

Let's introduce an abbreviation.

$A_1, A_2, \ldots \vDash B$	$\Leftrightarrow$	$A_1, A_2, \ldots$ logically entail B.
	$\Leftrightarrow$	There is no conceivable scenario in which $A_1, A_2, \ldots$ are all true while B is false, on any interpretation of the non-logical expressions.

The double-barred turnstile

Examples:

- $p \vee q, \neg q \models p$.
- $\forall x(Mx \to Ox), \exists x(Ax \to Mx) \models \exists x(Ax \to Ox)$.
- $\Diamond r, \Box(r \to w) \models \Diamond w$.

The double-barred turnstile

Do not confuse the turnstile $\models$ with the arrow $\to$!

- $\to$ is a symbol in the object language of propositional and predicate logic.
- $\models$ is a symbol in the meta-language in which we talk about the object language.
- $p \to q$ is a well-formed sentence of propositional logic; $p \models q$ is not.
- $p \to (q \to r)$ is well-formed, $p \to (q \models r)$ is gibberish.
- $p, q \models r$ is well-formed, $p, q \to r$ is gibberish.

The double-barred turnstile

Do not confuse the turnstile $\models$ with the arrow $\rightarrow$!

- $A \rightarrow B$ means that either A is false or B is true.
- $A \models B$ means that there is no conceivable scenario in which A is true while B is false, on any interpretation of the non-logical expressions.

The double-barred turnstile

A special case:

$\models B$ means that there is no conceivable scenario in which B is false, on any interpretation of the non-logical expressions.

(Here, '$\models$ B' is pronounced '*B* is (logically) valid'.)

The double-barred turnstile

A connection between $\to$ and $\models$:

$A \models B$ iff $\models A \to B$.

Think about why!

Modal logics

Modal logics

Whether $A_1, A_2, \ldots \models B$ does not depend on the interpretation of the non-logical expressions.

It does depend on the interpretation of the logical expressions.

Remember that the box means different things in different applications of modal logic.

Modal logics

Example: $\Box p \models p$

Is this plausible if the box means

- it could not have failed to be the case that?
- it is obligatory that?
- it is known that?
- it is mathematically provable that?
- Alice believes that?

There are different modal logics.

$\Box p \models_T p$
$\Box p \not\models_K p$