Logic 2: Modal Logic

Lecture 2

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Syntax of modal propositional logic

Syntax of modal propositional logic

The sentences of L_M are defined as follows.

1. Every sentence letter $p, q, r, \ldots$ is an L_M-sentence.
2. If A is an L_M-sentence, then so are $\neg A$, $\Diamond A$, and $\Box A$.
3. If A and B are L_M-sentences, then so are $(A \land B)$, $(A \lor B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.
4. Nothing else is an L_M-sentence.

The capital letters 'A', 'B', ...are not part of L_M. They are meta-linguistic variables to talk about arbitrary L_M-sentences.

Syntax of modal propositional logic

$\Box A \to A$ is a schema. Instances of this schema include

- $\Box p \to p$
- $\Box q \to q$
- $\Box(p \wedge q) \to (p \wedge q)$
- $\Box\Diamond p \to \Diamond p$
- $\Box\Diamond\Box\Diamond\neg p \to \Diamond\Box\Diamond\neg p$

Syntax of modal propositional logic

In applications, people often use other symbols instead of $\Box$ and $\Diamond$:

$$
\begin{aligned}
\mathsf{K}\, p &\Leftrightarrow \text{it is known that } p \\
\mathsf{K}_a\, p &\Leftrightarrow \text{agent } a \text{ knows that } p \\
\mathsf{B}_a\, p &\Leftrightarrow \text{agent } a \text{ believes that } p \\
\mathsf{F}\, p &\Leftrightarrow \text{it will be the case that } p \\
\mathsf{P}\, p &\Leftrightarrow \text{it was the case that } p \\
\mathsf{O}\, p &\Leftrightarrow \text{it is obligatory that } p \\
\mathsf{P}\, p &\Leftrightarrow \text{it is permitted that } p
\end{aligned}
$$

Translating from English

Translating from English

Translating from English into modal propositional logic isn't always easy.

- Try to paraphrase the original sentence in terms of 'necessary' and 'possible', and formalize these as $\Box$ and $\Diamond$, respectively.
- Read out the translated sentence and check if it says the same thing as the original: can one be true and the other false?

Translating from English

- Alice can’t leave the room.
- It is not possible that Alice leaves the room.
- $\neg\Diamond p$

- Alice mustn’t leave the room.
- It is necessary that Alice doesn’t leave the room.
- $\Box\neg p$
- Why would $\neg\Box p$ be wrong?

Translating from English

- Going to lectures is no guarantee that you'll do well in the exam.
- It is not necessary that if you go to lectures you will do well in the exam.
- $\neg\Box(g \to w)$
- $\Diamond(g \wedge \neg w)$

First steps in modal logic

First steps in modal logic

The main task when defining a logic is to specify which sentences entail which other sentences.

- Do $\Diamond r$ and $\Box(r \to w)$ entail $\Diamond w$?
- Does $\Box p$ entail p?
- ...

- $\Diamond r, \Box(r \to w) \models \Diamond w$?
- $\Box p \models p$?
- ...

First steps in modal logic

If $\Box p$ entails p, then plausibly any sentence of the form $\Box A$ entails the corresponding sentence A.

We write this like so: $\Box A \models A$

First steps in modal logic

If we use a box and a diamond, we always assume that

- $\Diamond A$ is equivalent to $\neg\Box\neg A$;
- $\Box A$ is equivalent to $\neg\Diamond\neg A$.

So we have, for example:

- $\neg\Box A \models \Diamond\neg A$
- $\neg\Diamond A \models \Box\neg A$

First steps in modal logic

We assume that ¬, ∧, ∨, →, ↔ have their standard meaning, given by the truth-tables.

We therefore have:

$$
\begin{aligned}
A \wedge B &\models A \\
\neg\neg A &\models A \\
A &\models B \to A \\
A, A \to B &\models B \\
&\dots
\end{aligned}
$$

First steps in modal logic

Remember:

$$A \models B \text{ iff } \models A \to B$$

More generally,

$$A_1, \ldots, A_n \models B \text{ iff } A_1, \ldots, A_{n-1} \models A_n \to B$$

First steps in modal logic

Instead of asking whether $A_1, \ldots, A_n$ entail B, we can ask whether $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$ is valid.

$$
\begin{aligned}
A_1, A_2, A_3 &\models B \\
A_1, A_2 &\models A_3 \rightarrow B \\
A_1 &\models A_2 \rightarrow (A_3 \rightarrow B) \\
&\models A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow B)) \\
&\models (A_1 \wedge A_2 \wedge A_3) \rightarrow B
\end{aligned}
$$

To specify which sentences entail which other sentences, it suffices to specify which sentences are logically valid.

Systems of modal logic

Systems of modal logic

In the early days of formal logic, the standard approach was to lay down some axioms and inference rules.

The Frege-Łukasiewicz axiomatization of propositional logic:

- **(A1)** $A \to (B \to A)$
- **(A2)** $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- **(A3)** $(\neg A \to \neg B) \to (B \to A)$
- **(MP)** From A and $A \to B$ one may infer B

All truth-functional tautologies are derivable from A1–A3 by MP.
There are no proofs from premises. To show that A entails B, you prove $A \to B$.

Systems of modal logic

Axiomatic derivations are often hard to find.

1.	$p \to ((p \to p) \to p)$	(A1)
2.	$(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$	(A2)
3.	$(p \to (p \to p)) \to (p \to p)$	(1, 2, MP)
4.	$p \to (p \to p)$	(A1)
5.	$p \to p$	(3, 4, MP)

Systems of modal logic

Kurt Gödel's (1933) axiomatization of the modal logic S4:

- **(PL)** The axioms of propositional logic
- **(K)** $\Box(A \to B) \to (\Box A \to \Box B)$
- **(T)** $\Box A \to A$
- **(4)** $\Box A \to \Box\Box A$
- **(MP)** From A and $A \to B$ one may infer B
- **(Nec)** From A one may infer $\Box A$

Gödel was interested in a reading of the box as constructive provability. He showed that S4 precisely captures the doctrines of "intuitionistic logic".

Systems of modal logic

A little earlier, C.I. Lewis had put forward a range of axiomatic modal systems, which he called S1–S5, with the aim of formalizing the concept of implication.

Guiding thought: A implies B iff it is impossible for A to be true and B false; i.e., iff $\neg\Diamond(A \wedge \neg B)$.

Lewis wasn't sure which principles should count as valid on the relevant understanding of 'possible'.

(Dis) $\Diamond(A \wedge B) \to \Diamond A$

(4) $\Diamond\Diamond A \to \Diamond A$

(5) $\Diamond A \to \Box\Diamond A$

Systems of modal logic

A wide range of interesting modal logics can be defined by adding axiom schemas to the system known as K:

(PL) The axioms of propositional logic

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

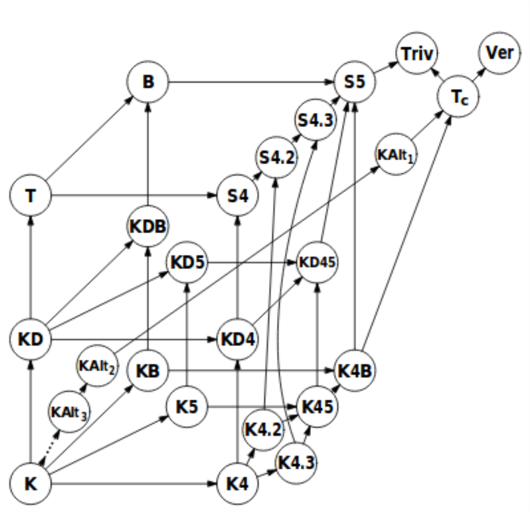
(MP) From A and $A \to B$ one may infer B

(Nec) From A one may infer $\Box A$

Any such logic is called normal.

Examples:

- S4 = K + **T** + **4**
- S5 = K + **T** + **5**


Triv
Ver
B
S5
Tc
S4.3
S4.2
KAlt1
T
S4
KDB
KD5
KD45
KD
KD4
KAlt2
KB
K4B
KAlt3
K5
K45
K4.2
K4.3
K
K4