Logic 2: Modal Logic

Lecture 3

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Review

Review

- We have added the box $\Box$ and the diamond $\Diamond$ to the language of propositional logic.
- $\Box$ and $\Diamond$ have different (intuitive) meanings in different applications of modal logic.
- Often, the box stands for some kind of necessity and the diamond for some kind of possibility.
- We always assume that the box and the diamond are duals, so that $\neg\Box A$ is equivalent to $\Diamond\neg A$ and $\neg\Diamond A$ is equivalent to $\Box\neg A$.

Review

- We would like to specify which L_M-sentences logically entail which other sentences.
- To this end, it suffices to specify which sentences are logically valid.
- For example, $\Box p$ entails p iff $\Box p \to p$ is valid.
- We need different conceptions of validity for different branches of modal logic.
- The classical axiomatic approach specifies the valid sentences by axioms and inference rules. A sentence is valid iff it follows from the axioms by the rules.

Review

- Candidate axioms include

$$\begin{aligned}
&(\mathbf{T}) \quad \Box A \to A \\
&(\mathbf{4}) \quad \Box A \to \Box\Box A \\
&(\mathbf{5}) \quad \Diamond A \to \Box\Diamond A \\
&(\mathbf{B}) \quad A \to \Box\Diamond A \\
&(\mathbf{G}) \quad \Diamond\Box A \to \Box\Diamond A \\
&(\mathbf{Ge}) \quad \Diamond(\Diamond\Diamond A \to \Box\Diamond B) \to (\Diamond\Diamond A \to \Box\Diamond B)
\end{aligned}$$

- It is often hard to tell intuitively whether a schema should count as valid or invalid.
- We need to get a better grip on the meaning of the modal operators.

Possible-worlds semantics

Possible-worlds semantics

A Leibnizian idea:

- Our world is one of many possible worlds.
- A sentence can be true at some worlds and false at others.
- ‘It is necessary that *p*’ says that *p* is true at all worlds.
- ‘It is possible that *p*’ says that *p* is true at some worlds.

Possible-worlds semantics

Consider the **4** schema $\Box A \to \Box\Box A$.

- Suppose $\Box A$ is true at our world.
- So A is true at all worlds.
- So $\Box A$ is true at all worlds.
- So $\Box\Box A$ is true at our world.

If we interpret the box as a quantifier over worlds, then $\Box A \to \Box\Box A$ is valid.

Possible-worlds semantics

All of these turn out to be valid:

(T) $\Box A \to A$

(4) $\Box A \to \Box\Box A$

(5) $\Diamond A \to \Box\Diamond A$

(B) $A \to \Box\Diamond A$

(G) $\Diamond\Box A \to \Box\Diamond A$

(Ge) $\Diamond(\Diamond\Diamond A \to \Box\Diamond B) \to (\Diamond\Diamond A \to \Box\Diamond B)$

Examples of invalid schemas:

(Triv) $\Box A \leftrightarrow A$

(M) $\Box\Diamond A \to \Diamond\Box A$

(GL) $\Box(\Box A \to A) \to \Box A$

Possible-worlds semantics

(NB: Next week we will alter our Leibnizian semantics so that e.g. $\Box A \to A$ can become invalid.)

The tree method

The tree method

The tree method (a.k.a. the method of analytic tableau) is a simple method to figure out whether a sentence or schema is valid or invalid.

The method works not only for modal propositional logic, but for many other logics, including classical propositional and predicate logic.

The tree method

Suppose we want to find out whether $p \to (q \to (r \vee p))$ is valid (in classical propositional logic).

We start by negating the target sentence:

$$1. \quad \neg(p \to (q \to (r \vee p))) \qquad (A)$$

Our aim is to derive a contradiction from this assumption.

The sentence has the form $\neg(A \to B)$.

We know that $A \to B$ is false only if A is true and B is false.

So assumption 1 entails p and $\neg(q \to (r \vee p))$.

We add these consequence below assumption 1.

The tree method

Target sentence: $p \to (q \to (r \lor p))$

1.	$\neg(p \to (q \to (r \lor p)))$	(A)
2.	p	(1)
3.	$\neg(q \to (r \lor p))$	(1)

Assumption 3 also has the form $\neg(A \to B)$.

We know that $A \to B$ is false only if A is true and B is false.

The tree method

Target sentence: $p \to (q \to (r \lor p))$

1.	$\neg(p \to (q \to (r \lor p)))$	(A)
2.	p	(1)
3.	$\neg(q \to (r \lor p))$	(1)
4.	q	(3)
5.	$\neg(r \lor p)$	(3)

Assumption 5 has the form $\neg(A \lor B)$.

We know that $A \lor B$ is false only if A and B are both false.

The tree method

Target sentence: $p \to (q \to (r \vee p))$

1.	$\neg(p \to (q \to (r \vee p)))$	(A)
2.	p	(1)
3.	$\neg(q \to (r \vee p))$	(1)
4.	q	(3)
5.	$\neg(r \vee p)$	(3)
6.	$\neg r$	(5)
7.	$\neg p$	(5)
	X	

Assumption 1 has led to a contradiction: 2 and 7.

Tree construction rules

1. To show that a sentence is valid, start the tree with its negation.
2. Then expand all nodes on the tree until no more nodes can be expanded.
3. To expand a non-negated node, you consider what the truth of the relevant sentence entails for the truth-values of its immediate parts. You then add these consequences to the tree.
4. To expand a negated node $\neg A$, you consider what the falsity of A entails for the truth-values of A's immediate parts. You then add these consequences to the tree.

The tree method

Target sentence: $(p \to q) \to ((q \to r) \to (p \to r))$

1. $\neg((p \to q) \to ((q \to r) \to (p \to r)))$ (A)

The tree method

Target sentence: $(p \to q) \to ((q \to r) \to (p \to r))$

1.	$\neg((p \to q) \to ((q \to r) \to (p \to r)))$	(A)
2.	$p \to q$	(1)
3.	$\neg((q \to r) \to (p \to r))$	(1)

The tree method

Target sentence: $(p \to q) \to ((q \to r) \to (p \to r))$

1.	$\neg((p \to q) \to ((q \to r) \to (p \to r)))$	(A)
2.	$p \to q$	(1)
3.	$\neg((q \to r) \to (p \to r))$	(1)
4.	$q \to r$	(3)
5.	$\neg(p \to r)$	(3)

The tree method

Target sentence: $(p \to q) \to ((q \to r) \to (p \to r))$

1.	$\neg((p \to q) \to ((q \to r) \to (p \to r)))$	(A)
2.	$p \to q$	(1)
3.	$\neg((q \to r) \to (p \to r))$	(1)
4.	$q \to r$	(3)
5.	$\neg(p \to r)$	(3)
6.	p	(5)
7.	$\neg r$	(5)

How do we expand assumptions 2 and 4?

The tree method

$$2. \qquad p \to q \qquad (1)$$

What can we infer from the truth of $p \to q$ about the truth-value of the immediate parts, p and q?

Either p is false or q is true.

We need to consider both possibilities.

The tree method

1.	$\neg((p \to q) \to ((q \to r) \to (p \to r)))$	(A)
2.	$p \to q$	(1)
3.	$\neg((q \to r) \to (p \to r))$	(1)
4.	$q \to r$	(3)
5.	$\neg(p \to r)$	(3)
6.	p	(5)
7.	$\neg r$	(5)

8.	$\neg p$	(2)	9.	q	(2)
	X				

The tree method

1. $\neg((p \to q) \to ((q \to r) \to (p \to r)))$ (A)
2. $p \to q$ (1)
3. $\neg(q \to r) \to (p \to r))$ (1)
4. $q \to r$ (3)
5. $\neg(p \to r)$ (3)
6. p (5)
7. $\neg r$ (5)

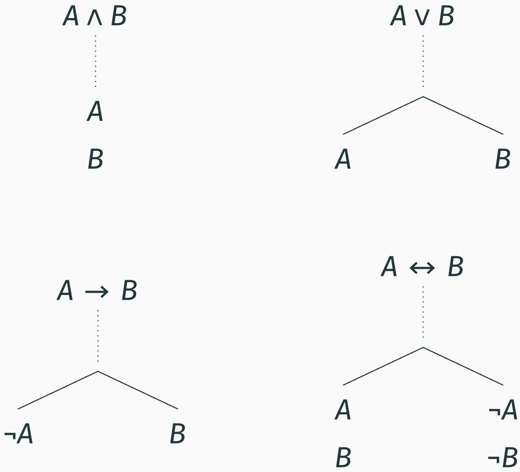
- 8. $\neg p$
  - x
- 9. q (2)
  - 10. $\neg q$
    - x
  - 11. r (4)
    - x

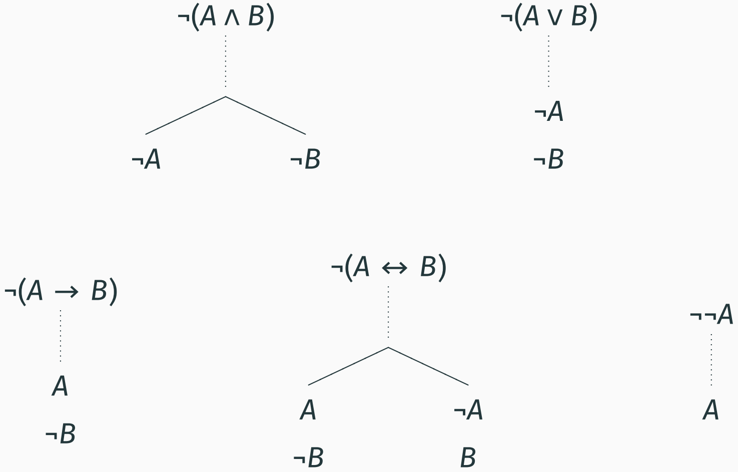
Tree construction rules

1. To show that a sentence is valid, start the tree with its negation.
2. Then expand all nodes on the tree until no more nodes can be expanded.
3. If a branch of a tree contains a sentence A and its negation $\neg A$, the branch is closed with an 'x'.
4. When a node is expanded, the new nodes are added to all open branches below the expanded node.

To keep your trees small, always expand non-branching nodes first.

The tree method




¬(A ∧ B)
¬A
¬B
¬(A ∨ B)
¬A
¬B
¬(A → B)
A
¬B
¬(A ↔ B)
A
¬B
¬A
B
¬¬A
A

Modal tree rules

Modal tree rules

Let's show that $\Box p \to p$ is valid.

Modal sentences are true or false relative to a world.

So our starting assumption is that $\Box p \to p$ is false at an arbitrary world w.

1.	$\neg(\Box p \to p)$	(w)	(A)

Our goal is to derive a contradiction from this assumption.

Modal tree rules

Target: $\Box p \to p$

1.	$\neg(\Box p \to p)$	(w)	(A)

If $\Box p \to p$ is false at w, then $\Box p$ is true at w and p is false at w.

Modal tree rules

Target: $\Box p \to p$

1.	$\neg(\Box p \to p)$	(w)	(A)
2.	$\Box p$	(w)	(1)
3.	$\neg p$	(w)	(1)

If $\Box p$ is true at w, then p is true at all worlds, including w.

Modal tree rules

Target: $\Box p \to p$

1.	$\neg(\Box p \to p)$	(w)	(A)
2.	$\Box p$	(w)	(1)
3.	$\neg p$	(w)	(1)
4.	p	(w)	(2)
	x		

p cannot be both true and false at w.

Modal tree rules

$\Box A \quad (\omega)$

$A \quad (v)$

↑

old

$\Diamond A \quad (\omega)$

$A \quad (v)$

↑

new

$\neg\Box A \quad (\omega)$

$\neg A \quad (v)$

↑

new

$\neg\Diamond A \quad (\omega)$

$\neg A \quad (v)$

↑

old

Modal tree rules

Target sentence: $p \to \Box\Diamond p$

1.	$\neg(p \to \Box\Diamond p)$	(w)	(A)

Modal tree rules

Target sentence: $p \to \Box\Diamond p$

1.	$\neg(p \to \Box\Diamond p)$	(w)	(A)
2.	p	(w)	(1)
3.	$\neg\Box\Diamond p$	(w)	(1)

Modal tree rules

Target sentence: $p \to \Box\Diamond p$

1.	$\neg(p \to \Box\Diamond p)$	(w)	(A)
2.	p	(w)	(1)
3.	$\neg\Box\Diamond p$	(w)	(1)
4.	$\neg\Diamond p$	(v)	(3)

Modal tree rules

Target sentence: $p \rightarrow \Box\Diamond p$

1.	$\neg(p \rightarrow \Box\Diamond p)$	(w)	(A)
2.	p	(w)	(1)
3.	$\neg\Box\Diamond p$	(w)	(1)
4.	$\neg\Diamond p$	(v)	(3)
5.	$\neg p$	(v)	(4)

Modal tree rules

Target sentence: $p \rightarrow \Box\Diamond p$

1.	$\neg(p \rightarrow \Box\Diamond p)$	(w)	(A)
2.	p	(w)	(1)
3.	$\neg\Box\Diamond p$	(w)	(1)
4.	$\neg\Diamond p$	(v)	(3)
5.	$\neg p$	(v)	(4)
6.	$\neg p$	(w)	(4)
	x		

Modal tree rules

Tree construction rules

1. To show that a sentence is valid, start the tree with the negation at world w.
2. Then expand all nodes on the tree until no more nodes can be expanded.
3. If a branch of a tree contains a sentence A and its negation $\neg A$ at the same world, the branch is closed with an 'x'.
4. Nodes of type $\Box A$ and $\neg\Diamond A$ can be expanded several times, once for each world on any branch to which the node belongs.
5. When a node of a type other than $\Box A$ and $\neg\Diamond A$ is expanded, the new nodes are added to all open branches below the expanded node, and the node is never expanded again.