Logic 2: Modal Logic

Lecture 7

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Recap

Recap

We have introduced a formal language ($\mathfrak{L}_M$) to reason about possibility, necessity, knowledge, belief, norms, time, and other non-truth-functional matters.

For each application, we need to clarify which $\mathfrak{L}_M$-sentences are valid, or entailed by which others.

- $\Box p \models p$?
- $\Box p \models \Diamond p$?
- $\Box p \models \Box\Box p$?
- $p \models \Box\Diamond p$?
- $\Diamond\Box p \models \Box\Diamond p$?

Recap

Many non-truth-functional operators can be analysed as (restricted) quantifiers over worlds or times.

It is historically necessary that $p \Leftrightarrow p$ is true at every possible world that we can bring about.

I know that $p \Leftrightarrow p$ is true at every possible world that is compatible with my evidence.

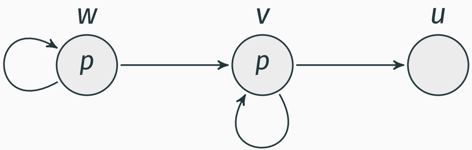
It is required that $p \Leftrightarrow p$ is true at every possible world in which the requirements are met.

It is always going to be the case that $p \Leftrightarrow p$ is true at every time after the present.

Recap

Many non-truth-functional operators can be analysed as (restricted) quantifiers over worlds or times.

$\Box p$ is true at $w \Leftrightarrow p$ is true at every world/time that is *accessible from w*.



Recap

We can define a logic by specifying formal properties of the accessibility relation.

If every world is accessible from itself then $\Box p \models p$.

If $\Box p \models p$ then every world is accessible from itself.

Recap

Schema		Condition On R
(T)	$\Box A \rightarrow A$	R is reflexive: every world in W is accessible from itself
(D)	$\Box A \rightarrow \Diamond A$	R is serial: every world in W can access some world in W
(B)	$A \rightarrow \Box\Diamond A$	R is symmetric: whenever wRv then vRw
(4)	$\Box A \rightarrow \Box\Box A$	R is transitive: whenever wRv and vRu, then wRu
(5)	$\Diamond A \rightarrow \Box\Diamond A$	R is euclidean: whenever wRv and wRu, then vRu
(G)	$\Diamond\Box A \rightarrow \Box\Diamond A$	R is convergent: whenever wRv and wRu, then there is some t such that vRt and uRt

Recap

Some aspects of the logic are the same no matter what we say about accessibility.

- $\Box A, \Box(A \rightarrow B) \models \Box B$
- $\Box(A \wedge B) \models \Box B$
- $\Diamond(A \vee B) \models \Diamond A \vee \Diamond B$
- $A \wedge B \models A$

Proofs

Proofs

Once we have specified a class of Kripke models (or frames), we have specified a logic.

But we haven't yet specified a method of proof for the logic.

Proofs

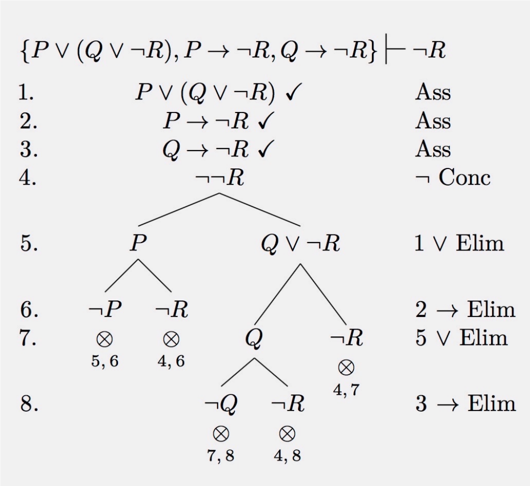
What is a proof of a sentence A?

- "A proof is list of sentences each of which is either an axiom or can be deduced from earlier sentences by one of the rules. A proof of A is such a list that ends with A."
- "A proof is a configuration of nodes – consisting of either an $\mathfrak{L}_M$-sentence with a world label or a sentence of the form $\omega R \nu$ – that conforms to the tree construction rules. A proof of A is such a configuration with starting node $\neg A$ (w) and in which all terminal nodes are marked as closed."
- ...

Proofs

1.	$((P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P)))$	by Ax2
2.	$(P \to ((P \to P) \to P))$	by Ax1
3.	$((P \to (P \to P)) \to (P \to P))$	from 2, 1 by MP
4.	$(P \to (P \to P))$	Ax1
5.	$(P \to P)$	from 4, 3 by MP

Proofs



$\{P \vee (Q \vee \neg R), P \rightarrow \neg R, Q \rightarrow \neg R\} \vdash \neg R$
1. $P \vee (Q \vee \neg R)$ ✓ Ass
2. $P \rightarrow \neg R$ ✓ Ass
3. $Q \rightarrow \neg R$ ✓ Ass
4. $\neg\neg R$ ¬ Conc
5. P $Q \vee \neg R$ 1 ∨ Elim
6. $\neg P$ $\neg R$ 2 → Elim
7. Q $\neg R$ 5 ∨ Elim
⊗ 5, 6
⊗ 4, 6
⊗ 4, 7
8. $\neg Q$ $\neg R$ 3 → Elim
⊗ 7, 8
⊗ 4, 8

Proofs

1	SHØW: $1 : \Box\varphi \rightarrow \Box\psi$	$[3, LCOND]$
2	$1 : \Box\varphi$	*ass.*
3	SHØW: $1 : \Box\psi$	$[k+1, LRED]$
4	$1 : \neg\Box\psi$	*ass.*
5	SHØW: $1.1 : \varphi \wedge \neg\psi$	$[i+1, LE_2]$
6	$1.1 : \neg\varphi \wedge \psi$	*ass.*
7	$1.1 : \neg\varphi$	$(6, L\alpha E)$
8	$1.1 : \psi$	$(6, L\alpha E)$
9	SHØW: $1.1 : \varphi \leftrightarrow \psi$	
	$\mathcal{D}[\sigma/1.\sigma]$	
i	$1.1 : \varphi$	$(8, 9)$
$i+1$	$\bot$	$(7, i, L\bot I)$
$i+2$	$1.1 : \varphi$	$(5, L\alpha E)$
$i+3$	$1.1 : \neg\psi$	$(5, L\alpha E)$
$i+4$	SHØW: $1.1 : \varphi \leftrightarrow \psi$	
	$\mathcal{D}[\sigma/1.\sigma]$	
k	$1.1 : \psi$	$(i+2, i+4)$
$k+1$	$\bot$	$(i+3, k, L\bot I)$

Proofs

1	$p \to q$	ass.
2	$q \to r$	ass.
3	p	ass.
4	$p \to q$	1, (rep.)
5	q	3, 4, $(\to E)$
6	$q \to r$	2, (rep.)
7	r	5, 6, $(\to E)$
8	$p \to r$	3–7 $(\to I)$
9	$(q \to r) \to (p \to r)$	2–8, $(\to I)$
10	$(p \to q) \to ((q \to r) \to (p \to r))$	1–9, $(\to I)$

Proofs

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \to (B \to C)}^{\,2} \qquad \dfrac{\overline{A \land B}^{\,1}}{A}}{B \to C} \qquad \dfrac{\overline{A \land B}^{\,1}}{B}}{C}}{A \land B \to C}\,{}^{1}}{(A \to (B \to C)) \to (A \land B \to C)}\,{}^{2}
$$

Proofs

$$\dfrac{}{B \vdash B}\,(I) \qquad \dfrac{}{C \vdash C}\,(I)$$

$$\dfrac{B \vdash B \qquad C \vdash C}{B \vee C \vdash B, C}\,(\vee L)$$

$$\dfrac{B \vee C \vdash B, C}{B \vee C \vdash C, B}\,(PR)$$

$$\dfrac{B \vee C \vdash C, B}{B \vee C, \neg C \vdash B}\,(\neg L) \qquad \dfrac{}{\neg A \vdash \neg A}\,(I)$$

$$\dfrac{B \vee C, \neg C \vdash B \qquad \neg A \vdash \neg A}{(B \vee C), \neg C, (B \to \neg A) \vdash \neg A}\,(\to L)$$

$$\dfrac{(B \vee C), \neg C, (B \to \neg A) \vdash \neg A}{(B \vee C), \neg C, ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(\wedge L_1)$$

$$\dfrac{(B \vee C), \neg C, ((B \to \neg A) \wedge \neg C) \vdash \neg A}{(B \vee C), ((B \to \neg A) \wedge \neg C), \neg C \vdash \neg A}\,(PL)$$

$$\dfrac{(B \vee C), ((B \to \neg A) \wedge \neg C), \neg C \vdash \neg A}{(B \vee C), ((B \to \neg A) \wedge \neg C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(\wedge L_2)$$

$$\dfrac{(B \vee C), ((B \to \neg A) \wedge \neg C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}{(B \vee C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(CL)$$

$$\dfrac{(B \vee C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}{((B \to \neg A) \wedge \neg C), (B \vee C) \vdash \neg A}\,(PL)$$

$$\dfrac{}{A \vdash A}\,(I)$$

$$\dfrac{A \vdash A}{\vdash \neg A, A}\,(\neg R)$$

$$\dfrac{\vdash \neg A, A}{\vdash A, \neg A}\,(PR)$$

$$\dfrac{\vdash A, \neg A \qquad ((B \to \neg A) \wedge \neg C), (B \vee C) \vdash \neg A}{((B \to \neg A) \wedge \neg C), (A \to (B \vee C)) \vdash \neg A, \neg A}\,(\to L)$$

Proofs

A proof is a finite syntactic object conforming to strict and mechanically testable rules.

Whatever method we use, we want it to have the following properties:

- **Soundness:** If a sentence is provable, then it is valid.
- **Completeness:** If a sentence is valid, then it is provable.

Soundness of K-trees

Soundness of K-trees

We have many concepts of validity, and different trees rules for each.

K-valid	K-rules
T-valid	K-rules + Reflexivity
D-valid	K-rules + Seriality
K4-valid	K-rules + Transitivity
S4-valid	K-rules + Reflexivity + Transitivity
S4.2-valid	K-rules + Reflexivity + Transitivity + Convergence
S5-valid	S5-rules
...	...

Soundness of K-trees

Let's show that the K-rules are sound for K-validity:

> If a K-tree for a target sentence closes, then that sentence is K-valid.

How could we show this?

Let's try a conditional proof:

- We assume there is a closed K-tree for some sentence A.
- We want to infer that A is K-valid.

Soundness of K-trees

- We assume there is a closed K-tree for some sentence A.
- We want to infer that A is K-valid. We want to infer that A is true at all worlds in all Kripke models.

Soundness of K-trees

- We assume there is a closed K-tree for some sentence A.
- We suppose that A is false at some world w in some Kripke model M.
- We want to derive a contradiction.

$$1. \quad \neg A \quad (w)$$

The first node on the tree is a correct statement about M.

Soundness of K-trees

- We assume there is a closed K-tree for some sentence A.
- We suppose that A is false at some world w in some Kripke model M.
- We want to derive a contradiction.

1.	$\neg(B \to C)$	(w)	
2.	B	(w)	(1)
3.	$\neg C$	(w)	(1)

After the first node is expanded, the new nodes are also correct statement about M.

Soundness of K-trees

- We assume there is a closed K-tree for some sentence A.
- We suppose that A is false at some world w in some Kripke model M.
- We want to derive a contradiction.

i. $B \vee C \quad (w)$

j. $B \quad (w)$ | k. $C \quad (w)$

After node i is expanded, the new node on at least one branch is also correct statement about M.

Soundness of K-trees

- We assume there is a closed K-tree for some sentence A.
- We suppose that A is false at some world w in some Kripke model M.
- We want to derive a contradiction.

In general, we can show this:

> *If all nodes on some branch of a tree are correct statements about M, and the branch is extended by the K-rules, then all nodes on at least one of the resulting branches are still correct statements about M.*

It follows that all nodes on some branch of the tree for A are correct statements about M.

Soundness of K-trees

- We assume there is a closed K-tree for some sentence A.
- We suppose that A is false at some world w in some Kripke model M.
- We want to derive a contradiction.
- The first node on the tree is a correct statement about M.
- Whenever a node on the tree is expanded, all nodes on at least one branch are all correct statements about M.
- But the tree is closed: every branch on the tree contains a contradictory pair

  n. B (v)

  m. $\neg B$ (v)

  These two nodes can't both be correct statements about M.

Completeness of K-trees

Completeness of K-trees

We have shown

Soundness

If a K-tree for a target sentence closes, then that sentence is K-valid.

Now we want to show

Completeness

If a sentence is K-valid, then there is a closed K-tree for the sentence.

Completeness of K-trees

Completeness

If a sentence is K-valid, then there is a closed K-tree for the sentence.

We will prove something even stronger:

- If a sentence is K-valid, then any fully expanded K-tree for the sentence is closed.

Equivalently:

- If a fully expanded K-tree does not close, then the target sentence is not K-valid.

Completeness of K-trees

If a fully expanded K-tree does not close, then the target sentence is not K-valid.

- We assume that a fully expanded K-tree for a target sentence A has an open branch.
- We want to infer that A is false at some world in some model.

We already know how to construct such a model: we can read it off from any open branch!

All we need to show is that our method for reading off a model from open branches always provides a countermodel for the target sentence.

Completeness of K-trees

Suppose there is an open branch on a fully expanded tree.

Let M be the model we read off from that branch.

We show that every node on the branch is a correct statement about M.

- The claim is obvious for sentence letters and negated sentence letters.
- Suppose $p \wedge q$ (w) is on the branch.
- Then p (w) and q (w) are on the branch.
- So p is true at w and q at w in M.
- So $p \wedge q$ is true at w in M.
- And so on.

Completeness of K-trees

Completeness

If a sentence is K-valid, then there is a closed K-tree for the sentence.

- We show that if there is a fully expanded but open K-tree for a sentence, then that sentence is not valid.
- We do this by showing that the model we can read off from an open branch on a fully expanded K-tree is always a countermodel for the target sentence.