# Logic 2: Modal Logic

Lecture 8

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# Axiomatic proofs

# Axiomatic proofs

An **axiomatic proof** is a list of sentences each of which is either

- an axiom of the proof system, or
- follows from earlier sentences by a rule of the proof system.

# Axiomatic proofs

**An axiomatic calculus for classical propositional logic:**

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**MP**) If $A$ and $A \to B$ occur on a proof, you may append $B$.

# Axiomatic proofs

**An axiomatic calculus for the modal logic K:**

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**MP**) If $A$ and $A \to B$ occur on a proof, you may append $B$.

(**Nec**) If $A$ occurs on a proof, you may append $\Box A$.

# Axiomatic proofs

The axiomatic method is useful to summarize a logical system.

Example: Which $\mathfrak{L}_M$-sentences are valid in the class of reflexive Kripke models?

1. The ones that can be proved with these 18 tree rules...
2. The ones that can be proved with these 26 natural deduction rules...
3. All propositional tautologies,
   all instances of $\Box(A \to B) \to (\Box A \to \Box B)$,
   all instances of $\Box A \to A$,
   and anything that can be derived from these by Modus Ponens and Necessitation.

# Axiomatic proofs

- A proof technique is **sound** if everything that's provable is valid.
- A proof technique is **complete** if everything that's valid is provable.

Let's prove soundness and completeness for the axiomatic calculus for K.

# Soundness

# Soundness

We want to show that **if**

- there is a derivation of a sentence $A$ from (A1)–(A3), (K) by (MP) and (Nec)

**then**

- $A$ is true at all worlds in all Kripke models.

# Soundness

We show that

1. Every instance of (A1), (A2), (A3), and (K) is K-valid.
2. If (MP) and (Nec) are applied to K-valid sentences, then the newly added sentence is also K-valid.

# Soundness

1. Every instance of $A \to (B \to A)$ is true at every world in every Kripke model.
2. Every instance of $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ is true at every world in every Kripke model.
3. Every instance of $(\neg A \to \neg B) \to (B \to A)$ is true at every world in every Kripke model.
4. Every instance of $\Box(A \to B) \to (\Box A \to \Box B)$ is true at every world in every Kripke model.

# Soundness

1. If $A \rightarrow B$ is true at every world in every Kripke model, and so is $A$, then $B$ is also true at every world in every Kripke model.
2. If $A$ is true at every world in every Kripke model, then $\Box A$ is true at every world in every Kripke model.

# Completeness

# Completeness

**To show:**

> If a sentence is K-valid, then it is provable from (A1)–(A3) and (K) by (MP) and (Nec).

For short: If *A* is K-valid, then *A* is K-provable.

The argument will be by contraposition:

We'll show that if *A* is not K-provable, then *A* is not K-valid.

# Completeness

To show: If $A$ is not K-provable, then $A$ is not K-valid.

- We assume that $A$ is not K-provable.
- We give a countermodel to show that $A$ is not K-valid.
- We use the same countermodel for every $A$: the **canonical model for K**.

# Completeness

To show: If $A$ is not K-provable, then $A$ is false at some world in the canonical model for K.

Canonical models are defined so that

(1) The worlds are sets of sentences.
(2) A sentence is true at a world iff it is a member of the world.
(3) Whenever a sentence is not provable, its negation is a member of some world.

# Completeness

(2) A sentence is true at a world iff it is a member of the world.

The set of sentences true at any world $w$ in any Kripke model $M$ is

- **maximal**: For every sentence $B$, the set contains either $B$ or $\neg B$;
- **K-consistent**: There is no sentence $B$ true at $w$ for which $\neg B$ is K-provable.

In any Kripke model $M$,

- $wRv$ only if $M, v \models A$ for all sentences $A$ for which $M, w \models \Box A$.

# Completeness

The **canonical model** $M_K$ for K is the Kripke model $(W, R, V)$, where

- $W$ is the set of all maximal K-consistent sets of $\mathfrak{L}_M$-sentences.
- $wRv$ iff $v$ contains every sentence $A$ for which $w$ contains $\Box A$.
- For every sentence letter $\rho$, $V(\rho)$ is the set of worlds in $W$ that contain $\rho$.

**Canonical Model Lemma**

$M_K, w \models A$ iff $A \in w$ (for any sentence $A$).

# Completeness

**To show:** If $A$ is K-valid then $A$ is K-provable.

**We show:** If $A$ is not K-provable then $A$ is false at some world in the canonical model for K.

1. Assume $A$ is not K-provable.
2. Then $\neg A$ is a member of some maximal K-consistent set. (Lindenbaum's Lemma)
3. So $\neg A \in w$ for some world $w$ in $M_{\mathsf{K}}$.
4. So $M_{\mathsf{K}}, w \models \neg A$, by the Canonical Model Lemma.
5. So $M_{\mathsf{K}}, w \not\models A$.

# More Completeness Proofs

**Completeness for S4:**

> *If a sentence is S4-valid, then it is provable from (A1)–(A3), (K), (T), and (4) by (MP) and (Nec).*

S4-valid means true at all worlds in all reflexive and transitive Kripke models.

**We show:**

> *If a sentence is not S4-provable then it is not S4-valid.*
>
> *If a sentence is not S4-provable then it is false at some world in the canonical model for S4, and this model is reflexive and transitive.*

## Canonical Model

The **canonical model** $M_{S4}$ for S4 is the Kripke model $\langle W, R, V \rangle$, where

- $W$ is the set of all maximal S4-consistent sets of $\mathfrak{L}_M$-sentences.
- $wRv$ iff $v$ contains every sentence $A$ for which $w$ contains $\Box A$.
- For every sentence letter $\rho$, $V(\rho)$ is the set of worlds in $W$ that contain $\rho$.

## Canonical Model Lemma

$M_{S4}, w \models A$ iff $A \in w$

# More Completeness Proofs

**To show:** If $A$ is S4-valid then $A$ is provable in the axiomatic calculus for S4.

**We show:** If $A$ is not S4-provable then $A$ is false at some world in the canonical model for S4.

1. Assume $A$ is not S4-provable.
2. Then $\neg A$ is a member of some maximal S4-consistent set. (Lindenbaum's Lemma)
3. So $\neg A \in w$ for some world $w$ in $M_{S4}$.
4. So $M_{S4}, w \models \neg A$, by the Canonical Model Lemma.
5. So $M_{S4}, w \not\models A$.

Still need to show that $M_{S4}$ is reflexive and transitive!

# The logic of provability

# The logic of provability

Our soundness and completeness proofs are informal.

But they could be formalised – as (say) axiomatic proofs – in the language of first-order predicate logic.

We would need some additional axioms about sets.

A suitable axiomatic calculus for proving soundness and completeness in modal logicanything is **ZFC**.

Non-logical axioms of ZFC:

1. $\exists x(x = \emptyset)$
2. $\forall xy(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$
3. $\forall x(x \neq \emptyset \rightarrow \exists y{\in}x \exists z{\in}x(y \notin z))$
4. $\forall \vec{y} z \exists v \forall x(x \in v \leftrightarrow x \in z \wedge \Phi(x, \vec{y}))$
5. $\forall xy \exists z(x \in z \wedge y \in z)$
6. $\forall x \exists y \forall z(z \in x \rightarrow z \subseteq y)$
7. $\forall \vec{z} v(\forall x{\in}v\, \exists! y \Phi(x, y, \vec{z}) \rightarrow \exists w \forall x{\in}v \exists y{\in}w\, \Phi(x, y, \vec{z}))$
8. $\exists x(\emptyset \in x \wedge \forall y{\in}x(y \cup \{y\} \in x))$
9. $\forall x \exists y \forall z(z \subseteq x \rightarrow z \in y)$
10. $\forall x \exists r$($r$ is a well-ordering of $x$)

# The logic of provability

In ZFC, one can prove

- that $2+2=4$
- that there are infinitely many prime numbers
- that $\Box(p \to q) \to (\Box p \to \Box q)$ is provable in the axiomatic calculus for K
- that the axiomatic calculus for K is sound and complete
- that $2+2=4$ is provable in ZFC
- ...

# The logic of provability

Let $\Box A$ mean that $A$ is provable in ZFC.

This box can be translated into the language of ZFC: There is a ZFC-predicate *Prov* such that a sentence $A$ is ZFC-provable iff ZFC can prove $Prov(\ulcorner A \urcorner)$.

ZFC can prove $\Box(2 + 2 = 4)$.

For any sentence $A$ that is ZFC-provable, ZFC can prove $\Box A$.

Any truth-functional consequence of ZFC-provable sentences is ZFC-provable.

ZFC can prove all instances of $\Box(A \to B) \to (\Box A \to \Box B)$.

The logic of ZFC-provability is an extension of K.

# The logic of provability

The logic of mathematical provability:

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

(T) $\Box A \to A$

(D) $\Box A \to \Diamond A$

(4) $\Box A \to \Box\Box A$

(5) $\Diamond A \to \Box\Diamond A$

(GL) $\Box(\Box A \to A) \to \Box A$

# The logic of provability

The logic of mathematical provability:

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

~~(T) $\Box A \to A$~~

~~(D) $\Box A \to \Diamond A$~~

(4) $\Box A \to \Box\Box A$

~~(5) $\Diamond A \to \Box\Diamond A$~~

(GL) $\Box(\Box A \to A) \to \Box A$

The system GL is sound and complete with respect to the class of finite, transitive, and irreflexive Kripke models.

(Completeness is hard to prove because the canonical model is infinite.)

# The logic of provability

(GL) $\Box(\Box A \to A) \to \Box A$

Suppose ZFC can prove $\neg\Box(2+2=5)$.

Then ZFC can prove $\Box(2+2=5) \to (2+2=5)$.

Then ZFC can prove $(2+2=5)$.

## A little History

In the 19th and early 20th century, powerful mathematical theories like ZFC were developed.

Some of these proposals turned out to be inconsistent.

(V) $\forall x(Fx \longleftrightarrow Gx) \longleftrightarrow \{x : Fx\} = \{x : Gx\}$

## A little History

David Hilbert tried to establish the consistency of axiomatised mathematical theories by unproblematic (finitary) methods.

An axiomatic calculus is *consistent* if one can't prove both $p$ and $\neg p$.

Proofs are finite mathematical objects.

## A little History

In 1931, Kurt Gödel showed that no consistent axiomatisable mathematical theory that is strong enough to prove elementary mathematical facts can prove its own consistency.

If ZFC can prove $\neg\Box(2+2=5)$ then ZFC can prove its own consistency.
If ZFC can prove its own consistency then ZFC is inconsistent.
If ZFC is inconsistent then ZFC can prove $(2+2=5)$.