

Logic 2: Modal Logic

Lecture 10

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Test topics:

- Translations
- Models
- Trees
- Comprehension questions

Epistemic Accessibility

Proposal 2

A world v is epistemically accessible for an agent at w (wRv) iff the agent's sense experiences and memory at v are the same as at w .

- Is R reflexive? (For all w , wRw)
- Is R transitive? (If wRv and vRu then wRu)
- Is R symmetric? (If wRv then vRw)

Yes. R is an equivalence relation.

Proposal 2

A world v is epistemically accessible for an agent at w (wRv) iff the agent's sense experiences and memory at v are the same as at w .

We get an S5 logic.

$$(K) \quad K(A \rightarrow B) \rightarrow (KA \rightarrow KB)$$

$$(T) \quad KA \rightarrow A$$

$$(4) \quad KA \rightarrow KKA$$

$$(B) \quad A \rightarrow KMA$$

$$(5) \quad MA \rightarrow KMA$$

Proposal 2

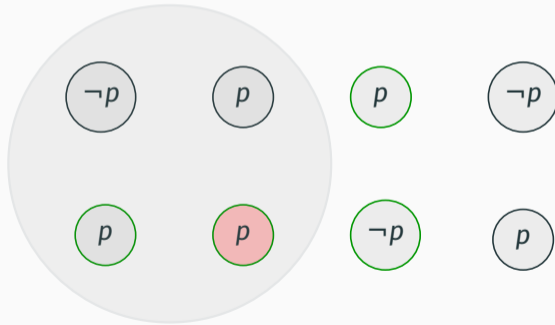
A world v is epistemically accessible for an agent at w (wRv) iff the agent's sense experiences and memory at v are the same as at w .

We also get scepticism about the external world.



Proposal 3 (Lewis 1996)

A world v is epistemically accessible for an agent at w (wRv) iff the agent's sense experiences and memory at v are the same as at w and v is not properly ignored



**Reflexivity, Seriality, Symmetry,
Transitivity, Euclidity**

Reflexivity, Seriality, Symmetry, Transitivity, Euclidity

Almost everyone wants the logic of knowledge to validate

$$(T) \quad KA \rightarrow A$$

So R should be reflexive. What about (D)?

$$(D) \quad KA \rightarrow MA$$

Reflexivity, Seriality, Symmetry, Transitivity, Euclidity

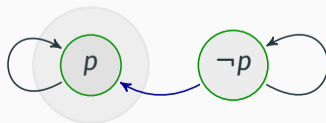
Should R be symmetric? Do we want (B) to come out valid?

(B) $A \rightarrow KMA$

Suppose you falsely believe $\neg p$.

- p is true.
- You believe that you know $\neg p$.
- You don't believe that you don't know $\neg p$.
- You don't know that you don't know $\neg p$.
- $\Box\neg\Box\neg p$ is false.
- $\Box\Diamond p$ is false.

Also, this would lead to skepticism.



Positive Introspection:

$$(4) KA \rightarrow KKA$$

Negative Introspection:

$$(5) MA \rightarrow KMA$$

(5) corresponds to euclidity. Euclidity and reflexivity entail symmetry. So philosophers mostly reject (5).

(4) corresponds to transitivity. It is controversial.

Multi-Modal Logic

If we want to talk about several agents, we need a multi-modal logic.

Definition

A **multi-modal Kripke model** consists of

- a non-empty set W ,
- a set of binary relation R_1, R_2, \dots, R_n on W , and
- a function V that assigns to each sentence letter of \mathcal{L}_M and each element of W a truth-value.

In epistemic logic, v is R_i -accessible from w iff v is compatible with the information agent i has at world w .

The **language of multi-modal propositional logic** has several boxes

$\Box_1, \Box_2, \dots, \Box_n$ and diamonds $\Diamond_1, \Diamond_2, \dots, \Diamond_n$.

$M, w \models \Box_i A$ iff $M, v \models A$ for all v such that $wR_i v$.

$M, w \models \Diamond_i A$ iff $M, v \models A$ for some v such that $wR_i v$.

As before, we write the boxes as 'K' and the diamonds as 'M'.

- $M_1 p$
- $K_1 p$
- $K_1 M_2 p$
- $K_1 p \rightarrow M_2 p$
- $K_1 p \rightarrow K_2 K_1 p$

Interaction principles

In multi-modal logics, we can impose constraints on individual accessibility relations:

- R_1 is reflexive
- R_2 is transitive
- etc.

but also on how different relations interact:

- if wR_1v then wR_2v
- if wR_1v then vR_2w
- if wR_1v and vR_2u then wR_3u
- etc.

Constraints on the interaction between accessibility relations correspond to **interaction schemas** that link different operators.

$$\diamond_1 A \rightarrow \diamond_2 A$$

$$\diamond_1 A \rightarrow \square_2 \diamond_1 A$$

etc.

An interaction principle for multi-agent knowledge:

$$K_1 K_2 A \rightarrow K_1 A$$

But this follows from the (T)-schema for K_2 :

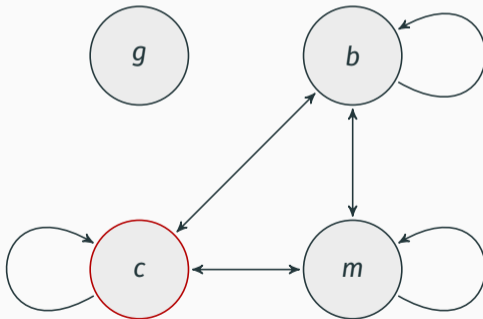
1. $K_2 A \rightarrow A$ (T)
2. $K_1(K_2 A \rightarrow A)$ (1, Nec)
3. $K_1(K_2 A \rightarrow A) \rightarrow (K_1 K_2 A \rightarrow K_1 A)$ (K)
4. $K_1 K_2 A \rightarrow K_1 A$ (2, 3, MP)

Knowledge and Belief

Knowledge and Belief

A belief state represents the world as being a certain way.

We can ask, for every possible world, whether it matches what an agent believes.



Is the **doxastic accessibility relation**

- reflexive ($\Box A \rightarrow A$)?
- serial ($\Box A \rightarrow \Diamond A$)?
- symmetric ($A \rightarrow \Box \Diamond A$)?
- transitive ($\Box A \rightarrow \Box \Box A$)?
- euclidean ($\Diamond A \rightarrow \Diamond \Box A$)?

If we accept seriality, transitivity, and euclidity, we get the logic KD45.

$M, w \models KA$ iff $M, v \models A$ for all v such that $wR_K v$

$M, w \models BA$ iff $M, v \models A$ for all v such that $wR_B v$

A plausible interaction principle: $KA \rightarrow BA$

What does this mean for R_B and R_K ?

$KA \rightarrow BA$

$\neg BA \rightarrow \neg KA$

$\Diamond_B A \rightarrow \Diamond_K A$

Whenever $wR_B v$ then $wR_K v$.

Candidate Interaction Principles for B and K:

(KB) $KA \rightarrow BA$

(PI) $BA \rightarrow KBA$

(NI) $\neg BA \rightarrow K\neg BA$

(SB) $BA \rightarrow BKA$

These entail

(B4) $BA \rightarrow BBA$

(B5) $\neg BA \rightarrow B\neg BA$

(KG) $MKA \rightarrow KMA$

Knowledge and Possibility

Let $\Diamond A$ mean that A is possible, in some circumstantial sense.

$M, w \models KA$ iff $M, v \models A$ for all v such that $wR_K v$

$M, w \models \Diamond A$ iff $M, v \models A$ for some v such that $wR_C v$

The verificationist **principle of knowability**: $A \rightarrow \Diamond KA$

1. Let p be any unknown truth. So $p \wedge \neg Kp$.
2. By the knowability principle, $\Diamond K(p \wedge \neg Kp)$.
3. $K(p \wedge \neg Kp)$ entails $Kp \wedge K\neg Kp$.
4. $K\neg Kp$ entails $\neg Kp$.
5. So Kp and $\neg Kp$.