Logic 2: Modal Logic

Lecture 5

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Recap

Recap

- We want to formalize reasoning about possibility, obligation, knowledge, past and future, etc.
- We have added new sentence operators $\Box$ and $\Diamond$ to the language of propositional logic.
- We have interpreted these as quantifiers over possible worlds.

Recap

- A basic model of $\mathfrak{L}_M$ consists of a non-empty set W and an interpretation function V that assigns truth-values to sentence letters at members of W.
- $\Box A$ is true at w in $\langle W, V \rangle$ iff A is true at all $v \in W$.
- $\Diamond A$ is true at w in $\langle W, V \rangle$ iff A is true at some $v \in W$.
- A sentence is valid if it is true at worlds in all models.

Recap

The following schemas come out valid.

(**K**) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(**T**) $\Box A \rightarrow A$

(**D**) $\Box A \rightarrow \Diamond A$

(**B**) $A \rightarrow \Box\Diamond A$

(**4**) $\Box A \rightarrow \Box\Box A$

(**5**) $\Diamond A \rightarrow \Box\Diamond A$

(**G**) $\Diamond\Box A \rightarrow \Box\Diamond A$

We don't always want these to be valid.

Kripke models

Kripke models

How can we avoid the validity of **T**, **D**, **B**, etc.?

Kripke (1963): replace these clauses

- $\Box A$ is true at w iff A is true at all worlds.
- $\Diamond A$ is true at w iff A is true at some world.

with those:

- $\Box A$ is true at w iff A is true at all worlds that are accessible from w.
- $\Diamond A$ is true at w iff A is true at some world that is accessible from w.

Also: Meredith and Prior 1956, Prior 1962, Kanger 1957, Montague 1955/1960, Hintikka 1957/1960/1961.

Kripke models

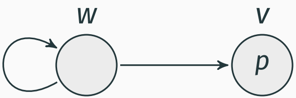
Definition: Kripke model

A Kripke model of $\mathfrak{L}_M$ is a triple $\langle W, R, V\rangle$ consisting of

- a non-empty set W,
- a binary relation R on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ and each element of W a truth-value.

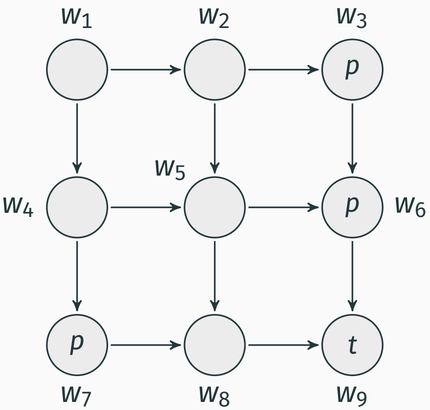
Intuitively: wRv iff v is possible relative to w.

Kripke models



- $\Diamond p$ is true at w.
- $\Diamond p$ is false at v.
- $\Box p$ is false at w.
- $\Box p$ is true at v. (Think $\neg\Diamond\neg p$.)
- $\Box\Diamond p$ is false at w.
- $\Box\Diamond p$ is true at v.

Kripke models



Where is $\Diamond t$ true? Where is $\Diamond\Box t$ true? Where is $\Diamond p$ true? Where is $\Box\Diamond p$ true?

The system K

The system K

Definition: K-valid

A sentence A is **K-valid** (for short, $\models_K A$) iff A is true at every world in every Kripke model.

Definition: S5-valid

A sentence A is **S5-valid** (for short, $\models_{S5} A$) iff A is true at every world in every basic model.

The set of all K-valid sentences is **system K**.

The set of all S5-valid sentences is **system S5**.

The system K

S5 was originally defined by axioms and rules:

(**PL**) The axioms of propositional logic

(**K**) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(**T**) $\Box A \rightarrow A$

(**5**) $\Diamond A \rightarrow \Box\Diamond A$

(**MP**) From A and $A \rightarrow B$ one may infer B

(**Nec**) From A one may infer $\Box A$

Turns out that a sentence is derivable from these axioms by these rules iff it is true at every world in every basic model.

The system K

K can also be defined by axioms and rules:

(**PL**) The axioms of propositional logic

(**K**) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(**MP**) From A and $A \rightarrow B$ one may infer B

(**Nec**) From A one may infer $\Box A$

A sentence is derivable from these axioms by these rules iff it is true at every world in every Kripke model.

The system K

Distinguish:

Schema K: $\Box(A \to B) \to (\Box A \to \Box B)$

System K: $\{S : S$ is true at all worlds in all Kripke models$\}$

System K contains many sentences that aren't instances of **K**.

- $p \lor \neg p$
- $\Box p \to \neg\Diamond\neg p$
- $\Box p \to \Box(p \lor q)$

The system K

Schema		S5	K
K	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓
T	$\Box A \to A$	✓	—
D	$\Box A \to \Diamond A$	✓	—
B	$A \to \Box\Diamond A$	✓	—
4	$\Box A \to \Box\Box A$	✓	—
5	$\Diamond A \to \Box\Diamond A$	✓	—
G	$\Diamond\Box A \to \Box\Diamond A$	✓	—

Tree rules for K

Tree rules for S5

$$\begin{array}{ccc} \Box A \quad (\omega) & \qquad & \Diamond A \quad (\omega) \\ \vdots & & \vdots \\ A \quad (\nu) & & A \quad (\nu) \\ \uparrow & & \uparrow \\ \text{old} & & \text{new} \end{array}$$

Tree rules for K

$$
\begin{array}{ll}
\Box A & (\omega) \\
\omega R \nu & \\
\vdots & \\
A & (\nu) \\
 & \uparrow \\
 & \text{old}
\end{array}
\qquad\qquad
\begin{array}{ll}
\Diamond A & (\omega) \\
\vdots & \\
\omega R \nu & \\
A & (\nu) \\
 & \uparrow \\
 & \text{new}
\end{array}
$$

Tree rules for K

Target sentence: $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

1.	$\neg(\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B))$	(w)	(Ass.)

Tree rules for K

Target sentence: $\Box(A \wedge B) \to (\Box A \wedge \Box B)$

1.	$\neg(\Box(A \wedge B) \to (\Box A \wedge \Box B))$	(w)	(Ass.)
2.	$\Box(A \wedge B)$	(w)	(1)
3.	$\neg(\Box A \wedge \Box B)$	(w)	(1)

Tree rules for K

Target sentence: $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

1.	$\neg(\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B))$	(w)	(Ass.)
2.	$\Box(A \wedge B)$	(w)	(1)
3.	$\neg(\Box A \wedge \Box B)$	(w)	(1)

4.	$\neg\Box A$	(w)	(3)	5.	$\neg\Box B$	(w)	(3)

Tree rules for K

Target sentence: $\Box(A \land B) \rightarrow (\Box A \land \Box B)$

1.	$\neg(\Box(A \land B) \rightarrow (\Box A \land \Box B))$	(w)	(Ass.)
2.	$\Box(A \land B)$	(w)	(1)
3.	$\neg(\Box A \land \Box B)$	(w)	(1)

Left branch:

4.	$\neg\Box A$	(w)	(3)
6.	wRv		(4)
7.	$\neg A$	(v)	(4)

Right branch:

5.	$\neg\Box B$	(w)	(3)

Tree rules for K

Target sentence: $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

1.	$\neg(\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B))$	(w)	(Ass.)
2.	$\Box(A \wedge B)$	(w)	(1)
3.	$\neg(\Box A \wedge \Box B)$	(w)	(1)

Left branch:

4.	$\neg\Box A$	(w)	(3)
6.	wRv		(4)
7.	$\neg A$	(v)	(4)
8.	$A \wedge B$	(v)	(2,6)

Right branch:

5.	$\neg\Box B$	(w)	(3)

Tree rules for K

Target sentence: $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

1.	$\neg(\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B))$	(w)	(Ass.)
2.	$\Box(A \wedge B)$	(w)	(1)
3.	$\neg(\Box A \wedge \Box B)$	(w)	(1)

Left branch:

4.	$\neg\Box A$	(w)	(3)
6.	wRv		(4)
7.	$\neg A$	(v)	(4)
8.	$A \wedge B$	(v)	(2,6)
9.	A	(v)	(8)
10.	B	(v)	(8)
	×		

Right branch:

5.	$\neg\Box B$	(w)	(3)

Tree rules for K

Target sentence: $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

1.	$\neg(\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B))$	(w)	(Ass.)
2.	$\Box(A \wedge B)$	(w)	(1)
3.	$\neg(\Box A \wedge \Box B)$	(w)	(1)

4.	$\neg\Box A$	(w)	(3)	5.	$\neg\Box B$	(w)	(3)
6.	wRv		(4)	11.	wRu		(5)
7.	$\neg A$	(v)	(4)	12.	$\neg B$	(u)	(5)
8.	$A \wedge B$	(v)	(2,6)	13.	$A \wedge B$	(u)	(2,11)
9.	A	(v)	(8)	14.	A	(u)	(13)
10.	B	(v)	(8)	15.	B	(u)	(13)
	x				x		