Logic 2: Modal Logic

Lecture 14

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Review

Review

Temporal operators:

- F: It will (at some point) be the case that
- G: It is always going to be the case that
- P: It was (at some point) the case that
- H: It has always been the case that

Review

Temporal Model

A **temporal model** consists of

- a non-empty set T (of "times"),
- a binary relation $<$ on T (the **precedence relation**),
- a function V that assigns to each sentence letter of $\mathfrak{L}_T$ and each member of T a truth-value (1 or 0).

Standard Temporal Semantics

(g) $M, t \models \mathrm{F}A$ iff $M, s \models A$ for some s such that $t < s$.

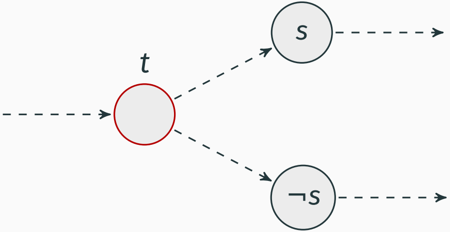
(h) $M, t \models \mathrm{G}A$ iff $M, s \models A$ for all s such that $t < s$.

(i) $M, t \models \mathrm{P}A$ iff $M, s \models A$ for some s such that $s < t$.

(j) $M, t \models \mathrm{H}A$ iff $M, s \models A$ for all s such that $s < t$.

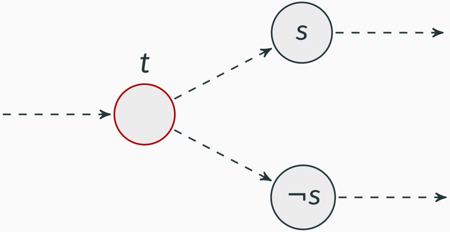
Branching time

Branching time



Is F s true at t? Yes.

Intuition: ‘There will be a sea battle’ is not true at t.



Standard semantics:

$M, t \models \mathsf{F} A$ iff $M, t' \models A$ for some t' such that $t < t'$
iff A is true at some future point on some history through t.

"Peircean" semantics (CTL):

$M, t \models \mathsf{F} A$ iff A is true at some future point on every history through t.

Is $\mathsf{F}\, s$ true at t in Peircean semantics? No.

Branching time

Standard semantics:

$M, t \models \mathsf{F} A$ iff A is true at some future point on some history through t.

“Peircean” semantics (CTL):

$M, t \models \mathsf{F} A$ iff A is true at some future point on every history through t.

We can factor out the quantification over histories.

$\Box A$: On every history (through the present point) ...
$\Diamond A$: On some history (through the present point) ...

$\Diamond \mathsf{F} A$: A is true at some future point on some history through t.
$\Box \mathsf{F} A$: A is true at some future point on every history through t.

Branching time

$\Box A$: On every history (through the present point) ...
$\Diamond A$: On some history (through the present point) ...
$\mathsf{F}A$: At some point in the future ...

How does this language work?

$M, t \models \Diamond A$ iff ?
$M, t \models \mathsf{F}A$ iff ?

"Ockhamist" semantics (CTL*):

$M, h, t \models \Diamond A$ iff $M, h', t \models A$ for some history h' that contains t.
$M, h, t \models \mathsf{F}A$ iff $M, h, t' \models A$ for some t' with $t < t'$.

Branching time

"Ockhamist" semantics (CTL*)

$M, h, t \models \Diamond A$ iff $M, h', t \models A$ for some history h' that contains t.

$M, h, t \models \mathsf{F} A$ iff $M, h, t' \models A$ for some t' with $t < t'$.

Truth is defined relative to three **parameters**: M, h, t.

Only M and t represent a scenario and an interpretation.

Ockhamism doesn't tell us which sentences are true in a given scenario under a given interpretation.

So it doesn't tell us which sentences are true in all scenarios under all interpretations.

Branching time

"Ockhamist" semantics (CTL*)

$$M, h, t \models \Diamond A \quad \text{iff } M, h', t \models A \text{ for some history } h' \text{ that contains } t.$$

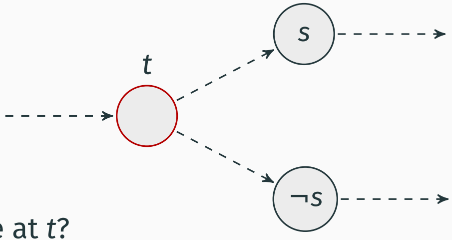
$$M, h, t \models \mathsf{F}A \quad \text{iff } M, h, t' \models A \text{ for some } t' \text{ with } t < t'.$$

Supervaluationism:

$M, t \models A$ iff $M, h, t \models A$ for every history h through t.

Branching time

$M, t \models A$ iff $M, h, t \models A$ for every history h through t.



Which of these are true at t?

- $\Diamond F s$
- $\Box F s$
- $F s$
- $\neg F s$
- $F s \vee \neg F s$

Branching time

Supervaluationist Ockhamism determines a **three-valued logic**. A sentence can be

- true
- false
- neither

The truth-value of a truth-functionally complex sentence at a scenario is not determined by the truth-value of the parts:

- Fs and $\neg Fs$ are neither true nor false, $Fs \vee \neg Fs$ is true.
- Fs and Fs are neither true nor false, $Fs \vee Fs$ is neither true nor false.

Branching time

In other three-valued logics, the truth-value of truth-functionally complex sentences is determined by the truth-values of the parts:

A	B	$A \vee B$
1	1	1
1	N	1
1	0	1
N	1	1
N	N	N
N	0	N
0	1	1
0	N	N
0	0	0

Two-Dimensional Modal Logic

Two-Dimensional Modal Logic

(*) We already knew yesterday that there would be a talk today.

Let Y mean ‘yesterday’.

(*) cannot be translated as $\mathrm{Y}\,\mathrm{K}\,p$.

- $\mathrm{Y}\,\mathrm{K}\,p$ is true today iff $\mathrm{K}\,p$ is true yesterday.
- If $\mathrm{K}\,p$ is true yesterday, then p is true yesterday.
- So $\mathrm{Y}\,\mathrm{K}\,p$ entails that p is true yesterday.

(*) We already knew yesterday that there would be a talk today.

Let N mean ‘today’ (or ‘now’).

Translation: $\mathrm{Y\,K\,N}\,p$.

How does this language work?

$M, t \models \mathrm{Y}\,A$ iff ?
$M, t \models \mathrm{N}\,A$ iff ?

This doesn’t work:

$M, t \models \mathrm{Y}\,A$ iff $M, t' \models A$ for the predecessor t' of t.
$M, t \models \mathrm{N}\,A$ iff $M, t \models A$.

We want $\mathrm{N}\,p$ to be true yesterday iff p is true today.

Two-Dimensional Semantics

$M, t_0, t \models \mathrm{Y}A$ iff $M, t_0, t' \models A$ for the predecessor t' of t.
$M, t_0, t \models \mathrm{N}A$ iff $M, t_0, t_0 \models A$.

Sentences are true relative to two times.

The first time parameter (t_0) holds fixed the time of the scenario.

Scenarios don't have two times. When is a sentence true at a time in a model?

Diagonalisation:
$M, t \models A$ iff $M, t, t \models A$.

An interesting consequence:

- $\models \mathrm{N}\, p \to p$
- $\not\models \mathrm{G}(\mathrm{N}\, p \to p)$

There are logical truths that may become false in the future!

Two-Dimensional Modal Logic

An old idea: apriority = analyticity = (unrestricted) necessity

‘I am here now’ appears to be

- a priori
- analytic
- contingent (not necessary)

‘It might have been that I am not here now.’

When we evaluate ‘I am here now’ at a counterfactual world *w*, ‘here’ and ‘now’ refer to my location *in the actual world*.