# Logic 2: Modal Logic

## Lecture 7

Wolfgang Schwarz

7 October 2019

University of Edinburgh

# Review

## Review

We have introduced a formal language ($\mathfrak{L}_M$) to reason about possibility, necessity, knowledge, belief, norms, time, and other non-truth-functional matters.

Suppose we take the box to formalize 'I know that'.

$\Diamond A$ always means $\neg\Box\neg A$.

We need to specify which $\mathfrak{L}_M$-sentences should count as valid on the given interpretation of the box.

## Review

- Does $\Box p$ entail $p$?
- Does $\Box(p \wedge q)$ entail $\Box p$?
- Does $\Box p$ entail $\Box\Box p$?
- Does $\Box p$ entail $\Diamond\Box p$?
- Does $\Diamond\Box p$ entail $\Box\Diamond p$?

# Review

- Is $\Box p \to p$ valid?
- Is $\Box(p \land q) \to \Box p$ valid?
- Is $\Box p \to \Box\Box p$ valid?
- Is $\Box p \to \Diamond\Box p$ valid?
- Is $\Diamond\Box p \to \Box\Diamond p$ valid?

# Review

Informally, a sentence is (logically) valid iff it is true in all conceivable scenarios, under every interpretation of the non-logical vocabulary.

$$\Box p \rightarrow p$$

We make this precise by using formal models to represent conceivable scenarios together with an interpretation of the non-logical vocabulary.

A sentence is valid iff it is true at all models.

Our models typically involve possible worlds.

# Review

Many non-truth-functional operators can be analysed as (restricted) quantifiers over worlds or times.

- I know that $p \iff p$ is true in every possible world that is compatible with my evidence.
- It is physically necessary that $p \iff p$ is true in every possible world that is compatible the laws of nature.
- It is required that $p \iff p$ is true in every possible world in which the requirements are met.
- It is always going to be the case that $p \iff p$ is true at all times after the present.

# Review

A Kripke model specifies a set $W$ (which we often interpret as possible worlds) and an accessibility relation on $W$. It also specifies the truth-value of every sentence letter at every world.

$\Box A$ is true at a world $w$ in a Kripke model $M$ iff $A$ is true at all worlds in $M$ that are accessible from $w$.

I know that $p$ is true iff $p$ is true in every possible world that is compatible with my evidence.

If the box is interpreted as 'I know that', $wRv$ therefore means that $v$ is compatible with my evidence at $w$.

# Review

What do Kripke models look like in which $vRw$ means that $v$ is compatible with my evidence at $w$?

The accessibility relation is

- plausibly reflexive: every world is accessible from itself;
- possibly transitive: if $v$ is accessible from $w$ and $u$ is accessible from $v$ then $u$ is accessible from $w$;
- not symmetric: if $v$ is accessible from $w$ then $w$ is not always accessible from $v$.
- ...

These conditions on the accessibility relation affect the resulting logic (the class of valid sentences).

## Review

If we assume that $R$ is reflexive (in every relevant Kripke model), then $\Box A \rightarrow A$ is valid.

| Schema | | Condition On $R$ |
|---|---|---|
| **T** | $\Box A \rightarrow A$ | $R$ is reflexive: every world in $W$ is accessible from itself |
| **D** | $\Box A \rightarrow \Diamond A$ | $R$ is serial: every world in $W$ can access some world in $W$ |
| **B** | $A \rightarrow \Box\Diamond A$ | $R$ is symmetric: whenever $wRv$ then $vRw$ |
| **4** | $\Box A \rightarrow \Box\Box A$ | $R$ is transitive: whenever $wRv$ and $vRu$, then $wRu$ |
| **5** | $\Diamond A \rightarrow \Box\Diamond A$ | $R$ is euclidean: whenever $wRv$ and $wRu$, then $vRu$ |
| **G** | $\Diamond\Box A \rightarrow \Box\Diamond A$ | $R$ is convergent: whenever $wRv$ and $wRu$, then there is some $t$ such that $vRt$ and $uRt$ |

# Validity and Provability

# Validity and Provability

Once we have specified the relevant class of Kripke models, we have specified which sentences are valid.

We have specified a logic (of knowledge, obligation, time, etc.).

How do you **show** that a sentence is valid?

By giving a proof.

# Validity and Provability

There are many proof systems.

| | | |
|---|---|---|
| 1 | $p \to q$ | ass. |
| 2 | $q \to r$ | ass. |
| 3 | $p$ | ass. |
| 4 | $p \to q$ | 1, (rep.) |
| 5 | $q$ | 3, 4, $(\to E)$ |
| 6 | $q \to r$ | 2, (rep.) |
| 7 | $r$ | 5, 6, $(\to E)$ |
| 8 | $p \to r$ | 3–7 $(\to I)$ |
| 9 | $(q \to r) \to (p \to r)$ | 2–8, $(\to I)$ |
| 10 | $(p \to q) \to ((q \to r) \to (p \to r))$ | 1–9, $(\to I)$ |

There are many proof systems.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to (B \to C)}\,{}^{2} \qquad \dfrac{\dfrac{}{A \land B}\,{}^{1}}{A}}{B \to C} \qquad \dfrac{\dfrac{}{A \land B}\,{}^{1}}{B}}{\dfrac{C}{A \land B \to C}\,{}^{1}}}{(A \to (B \to C)) \to (A \land B \to C)}\,{}^{2}$$

# Validity and Provability

There are many proof systems.

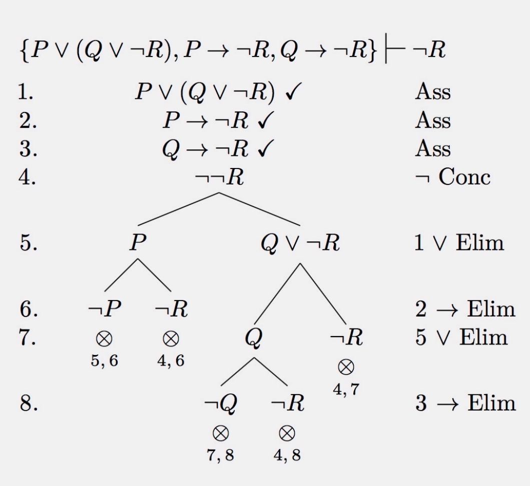
| | | |
|---|---|---|
| 1 | SHØW: $1 : \Box\varphi \rightarrow \Box\psi$ | $[3, LCOND]$ |
| 2 | $1 : \Box\varphi$ | $ass.$ |
| 3 | SHØW: $1 : \Box\psi$ | $[k+1, LRED]$ |
| 4 | $1 : \neg\Box\psi$ | $ass.$ |
| 5 | SHØW: $1.1 : \varphi \wedge \neg\psi$ | $[i+1, LE_2]$ |
| 6 | $1.1 : \neg\varphi \wedge \psi$ | $ass.$ |
| 7 | $1.1 : \neg\varphi$ | $(6, L\alpha E)$ |
| 8 | $1.1 : \psi$ | $(6, L\alpha E)$ |
| 9 | SHØW: $1.1 : \varphi \leftrightarrow \psi$ | |
| | $\mathcal{D}[\sigma/1.\sigma]$ | |
| $i$ | $1.1 : \varphi$ | $(8, 9)$ |
| $i+1$ | $\bot$ | $(7, i, L\bot I)$ |
| $i+2$ | $1.1 : \varphi$ | $(5, L\alpha E)$ |
| $i+3$ | $1.1 : \neg\psi$ | $(5, L\alpha E)$ |
| $i+4$ | SHØW: $1.1 : \varphi \leftrightarrow \psi$ | |
| | $\mathcal{D}[\sigma/1.\sigma]$ | |
| $k$ | $1.1 : \psi$ | $(i+2, i+4)$ |
| $k+1$ | $\bot$ | $(i+3, k, L\bot I)$ |

# Validity and Provability

There are many proof systems.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$A \vdash A$}
\RightLabel{$(\neg R)$}
\UnaryInfC{$\vdash \neg A, A$}
\RightLabel{(PR)}
\UnaryInfC{$\vdash A, \neg A$}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$B \vdash B$}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$C \vdash C$}
\RightLabel{$(\vee L)$}
\BinaryInfC{$B \vee C \vdash B, C$}
\RightLabel{(PR)}
\UnaryInfC{$B \vee C \vdash C, B$}
\RightLabel{$(\neg L)$}
\UnaryInfC{$B \vee C, \neg C \vdash B$}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$\neg A \vdash \neg A$}
\RightLabel{$(\rightarrow L)$}
\BinaryInfC{$(B \vee C), \neg C, (B \rightarrow \neg A) \vdash \neg A$}
\RightLabel{$(\wedge L_1)$}
\UnaryInfC{$(B \vee C), \neg C, ((B \rightarrow \neg A) \wedge \neg C) \vdash \neg A$}
\RightLabel{(PL)}
\UnaryInfC{$(B \vee C), ((B \rightarrow \neg A) \wedge \neg C), \neg C \vdash \neg A$}
\RightLabel{$(\wedge L_2)$}
\UnaryInfC{$(B \vee C), ((B \rightarrow \neg A) \wedge \neg C), ((B \rightarrow \neg A) \wedge \neg C) \vdash \neg A$}
\RightLabel{(CL)}
\UnaryInfC{$(B \vee C), ((B \rightarrow \neg A) \wedge \neg C) \vdash \neg A$}
\RightLabel{(PL)}
\UnaryInfC{$((B \rightarrow \neg A) \wedge \neg C), (B \vee C) \vdash \neg A$}
\RightLabel{$(\rightarrow L)$}
\BinaryInfC{$((B \rightarrow \neg A) \wedge \neg C), (A \rightarrow (B \vee C)) \vdash \neg A, \neg A$}
\end{prooftree}
$$

There are many proof systems.



# Validity and Provability

There are many proof systems.

| | | |
|---|---|---|
| 1. | $((P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P)))$ | by Ax2 |
| 2. | $(P \to ((P \to P) \to P))$ | by Ax1 |
| 3. | $((P \to (P \to P)) \to (P \to P))$ | from 2, 1 by MP |
| 4. | $(P \to (P \to P))$ | Ax1 |
| 5. | $(P \to P)$ | from 4, 3 by MP |

# Validity and Provability

Main types of proof system:

- Natural Deduction
- Trees (Tableaux)
- Axiomatic
- Sequent

The choice is largely a matter of taste.

# Validity and Provability

Whatever method we use, we want it to have the following properties:

- **Soundness**: If a sentence is provable, then it is valid.
- **Completeness**: If a sentence is valid, then it is provable.

# Soundness of K-trees

# Soundness of K-trees

We have many concepts of validity, and different trees rules for each.

| | |
|---|---|
| K-valid | K-rules |
| T-valid | K-rules + Reflexivity |
| D-valid | K-rules + Seriality |
| K4-valid | K-rules + Transitivity |
| S4-valid | K-rules + Reflexivity + Transitivity |
| S4.2-valid | K-rules + Reflexivity + Transitivity + Convergence |
| S5-valid | S5-rules |
| ... | ... |

# Soundness of K-trees

Let's show that the K-rules are sound for K-validity:

> If a K-tree for a target sentence closes, then that sentence is K-valid.

How could we show this?

Let's try a conditional proof:

- We assume there is a closed K-tree for some sentence $A$.
- We want to infer that $A$ is K-valid.

# Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We want to infer that $A$ is K-valid. We want to infer that $A$ is true at all worlds in all Kripke models.

The guiding idea behind the tree method is to assume that the target sentence is false at some world ($w$) in some model, and derive a contradiction.

$$1. \quad \neg A \quad (w)$$

We are going to show that if a tree for $A$ closes, then the hypothesis that $A$ is false at some world in some Kripke model leads to contradiction.

## Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.

$$1. \quad \neg A \quad (w)$$

The first node on the tree is a correct statement about $M$.

# Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.

| | | | |
|---|---|---|---|
| 1. | $\neg(B \to C)$ | $(w)$ | |
| 2. | $B$ | $(w)$ | (1) |
| 3. | $\neg C$ | $(w)$ | (1) |

After the first node is expanded, the new nodes are also correct statement about $M$.

# Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.

i. $B \lor C \quad (w)$

j. $B \quad (w)$ k. $C \quad (w)$

After node i is expanded, the new node on at least one branch is also correct statement about $M$.

# Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.

In general, we can show this:

> If all nodes on some branch of a tree are correct statements about $M$, and the branch is extended by the K-rules, then all nodes on at least one of the resulting branches are still correct statements about $M$.

It follows that all nodes on some branch of the tree for $A$ are correct statements about $M$.

# Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.
- The first node on the tree is a correct statement about $M$.
- Whenever a node on the tree is expanded, all nodes on at least one branch are all correct statements about $M$.
- But the tree is closed: every branch on the tree contains a contradictory pair

| | | |
|---|---|---|
| n. | $B$ | $(\upsilon)$ |
| m. | $\neg B$ | $(\upsilon)$ |

These two nodes can't both be correct statements about $M$.

# Completeness of K-trees

# Completeness of K-trees

We have shown

**Soundness**

If a K-tree for a target sentence closes, then that sentence is K-valid.

Now we want to show

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

# Completeness of K-trees

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

We will prove something even stronger:

- If a sentence is K-valid, then any fully expanded K-tree for the sentence is closed.

Equivalently:

- If a fully expanded K-tree does not close, then the target sentence is not K-valid.

# Completeness of K-trees

> If a fully expanded K-tree does not close, then the target sentence is not K-valid.

- We assume that a fully expanded K-tree for a target sentence $A$ has an open branch.
- We want to infer that $A$ is false at some world in some model.

We already know how to construct such a model: we can read it off from any open branch!

All we need to show is that our method for reading off a model from open branches always provides a countermodel for the target sentence.

# Completeness of K-trees

Suppose there is an open branch on a fully expanded tree.

Let $M$ be the model we read off from that branch.

We show that every node on the branch is a correct statement about $M$.

- The claim is obvious for sentence letters and negated sentence letters.
- Suppose $p \wedge q$ $(w)$ is on the branch.
- Then $p$ $(w)$ and $q$ $(w)$ are on the branch.
- So $p$ is true at $w$ and $q$ at $w$ in $M$.
- So $p \wedge q$ is true at $w$ in $M$.
- And so on.

# Completeness of K-trees

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

- We show that if there is a fully expanded but open K-tree for a sentence, then that sentence is not valid.
- We do this by showing that the model we can read off from an open branch on a fully expanded K-tree is always a countermodel for the target sentence.