Logic 2: Modal Logic

Lecture 7

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Review
We have introduced a formal language ($\mathcal{L}_M$) to reason about possibility, necessity, knowledge, belief, norms, time, and other non-truth-functional matters.

Suppose we take the box to formalize ‘I know that’.

◊$A$ always means $\neg\Box\neg A$.

We need to specify which $\mathcal{L}_M$-sentences should count as valid on the given interpretation of the box.
• Does □p entail p?
• Does □(p \land q) entail □p?
• Does □p entail □□p?
• Does □p entail ◊□p?
• Does ◊□p entail □◊p?
• Does ◊□p entail □◊p?
• Is $\Box p \rightarrow p$ valid?
• Is $\Box(p \land q) \rightarrow \Box p$ valid?
• Is $\Box p \rightarrow \Box\Box p$ valid?
• Is $\Box p \rightarrow \Diamond\Box p$ valid?
• Is $\Diamond\Box p \rightarrow \Box\Diamond p$ valid?
Informally, a sentence is (logically) valid iff it is true in all conceivable scenarios, under every interpretation of the non-logical vocabulary.

$$\Box p \rightarrow p$$

We make this precise by using formal models to represent conceivable scenarios together with an interpretation of the non-logical vocabulary.

A sentence is valid iff it is true at all models.

Our models typically involve possible worlds.
Many non-truth-functional operators can be analysed as (restricted) quantifiers over worlds or times.

- **I know that** $p \leftrightarrow p$ is true in every possible world that is compatible with my evidence.
- **It is physically necessary that** $p \leftrightarrow p$ is true in every possible world that is compatible the laws of nature.
- **It is required that** $p \leftrightarrow p$ is true in every possible world in which the requirements are met.
- **It is always going to be the case that** $p \leftrightarrow p$ is true at all times after the present.
A Kripke model specifies a set $W$ (which we often interpret as possible worlds) and an accessibility relation on $W$. It also specifies the truth-value of every sentence letter at every world.

$\square A$ is true at a world $w$ in a Kripke model $M$ iff $A$ is true at all worlds in $M$ that are accessible from $w$.

I know that $p$ is true iff $p$ is true in every possible world that is compatible with my evidence.

If the box is interpreted as ‘I know that’, $wRv$ therefore means that $v$ is compatible with my evidence at $w$. 
What do Kripke models look like in which \(vRw\) means that \(v\) is compatible with my evidence at \(w\)?

The accessibility relation is

- plausibly reflexive: every world is accessible from itself;
- possibly transitive: if \(v\) is accessible from \(w\) and \(u\) is accessible from \(v\) then \(u\) is accessible from \(w\);
- not symmetric: if \(v\) is accessible from \(w\) then \(w\) is not always accessible from \(v\).
- ...

These conditions on the accessibility relation affect the resulting logic (the class of valid sentences).
If we assume that $R$ is reflexive (in every relevant Kripke model), then $\square A \rightarrow A$ is valid.

<table>
<thead>
<tr>
<th>Schema</th>
<th>Condition On $R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\square A \rightarrow A$</td>
</tr>
<tr>
<td>D</td>
<td>$\square A \rightarrow \Diamond A$</td>
</tr>
<tr>
<td>B</td>
<td>$A \rightarrow \square \Diamond A$</td>
</tr>
<tr>
<td>4</td>
<td>$\square A \rightarrow \square \square A$</td>
</tr>
<tr>
<td>5</td>
<td>$\Diamond A \rightarrow \square \Diamond A$</td>
</tr>
<tr>
<td>G</td>
<td>$\Diamond \square A \rightarrow \square \Diamond A$</td>
</tr>
</tbody>
</table>

- $R$ is reflexive: every world in $W$ is accessible from itself
- $R$ is serial: every world in $W$ can access some world in $W$
- $R$ is symmetric: whenever $wRv$ then $vRw$
- $R$ is transitive: whenever $wRv$ and $vRu$, then $wRu$
- $R$ is euclidean: whenever $wRv$ and $wRu$, then $vRu$
- $R$ is convergent: whenever $wRv$ and $wRu$, then there is some $t$ such that $vRt$ and $uRt$
Validity and Provability
Validity and Provability

Once we have specified the relevant class of Kripke models, we have specified which sentences are valid.

We have specified a logic (of knowledge, obligation, time, etc.).

How do you **show** that a sentence is valid?

By giving a proof.
There are many proof systems.

1. \( p \rightarrow q \)
2. \( q \rightarrow r \)
3. \( p \)
4. \( p \rightarrow q \)
5. \( q \)
6. \( q \rightarrow r \)
7. \( r \)
8. \( p \rightarrow r \)
9. \( (q \rightarrow r) \rightarrow (p \rightarrow r) \)
10. \( (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) \)
There are many proof systems.

\[
\begin{align*}
A \rightarrow (B \rightarrow C) & \quad \text{2} \\
\frac{A \land B}{A} & \quad \text{1} \\
\frac{B \rightarrow C}{C} & \quad \text{1} \\
\frac{C}{A \land B \rightarrow C'} & \quad \text{1} \\
\frac{(A \rightarrow (B \rightarrow C')) \rightarrow (A \land B \rightarrow C')}{(A \land B \rightarrow C')} & \quad \text{2}
\end{align*}
\]
Validity and Provability

There are many proof systems.

1: □φ \rightarrow □ψ

SHØW: 1: □φ

SHØW: 1: □ψ

1: \neg □ψ

SHØW: 1.1: \varphi \wedge \neg \psi

1.1: \neg \varphi \wedge \psi

1.1: \neg \varphi

1.1: \psi

SHØW: 1.1: \varphi \leftrightarrow \psi

\frac{\Delta[\sigma/1.\sigma]}{}

1.1: \varphi

\bot

(i + 1, LRE_D)

\bot

(i + 2, LRE_D)

(i + 3, LRE_D)

\bot

(i + 2, i + 4)

(i + 3, k, L_\bot I)
Validity and Provability

There are many proof systems.

\[
\begin{align*}
\frac{B \vdash B \quad C \vdash C}{B \lor C \vdash B, C} \quad \text{(\lor L)} \\
\frac{B \lor C \vdash C, B}{B \lor C \vdash C, B} \quad \text{(PR)} \\
\frac{B \lor C \vdash C, B}{B \lor C, \neg C \vdash B} \quad \frac{\neg A \vdash \neg A}{\neg A \vdash \neg A} \quad \text{(\lor I)} \\
\frac{(B \lor C), \neg C, (B \rightarrow \neg A) \vdash \neg A}{(\rightarrow L)} \\
\frac{(B \lor C), \neg C, ((B \rightarrow \neg A) \land \neg C) \vdash \neg A}{(\land L_1)} \\
\frac{(B \lor C), (B \rightarrow \neg A) \land \neg C \vdash \neg C}{(PL)} \\
\frac{(B \lor C), ((B \rightarrow \neg A) \land \neg C), \neg C \vdash \neg A}{(\land L_2)} \\
\frac{(B \lor C), ((B \rightarrow \neg A) \land \neg C), (B \rightarrow \neg A) \land \neg C) \vdash \neg A}{(CL)} \\
\frac{(B \lor C), ((B \rightarrow \neg A) \land \neg C), (B \rightarrow \neg A) \land \neg C) \vdash \neg A}{(PL)} \\
\frac{(B \rightarrow \neg A) \land \neg C), (B \lor C) \vdash \neg A}{(\rightarrow L)}
\end{align*}
\]
There are many proof systems.

\[
\{ P \lor (Q \lor \lnot R), P \rightarrow \lnot R, Q \rightarrow \lnot R \} \models \lnot R
\]

1. \( P \lor (Q \lor \lnot R) \) \( \checkmark \) Ass
2. \( P \rightarrow \lnot R \) \( \checkmark \) Ass
3. \( Q \rightarrow \lnot R \) \( \checkmark \) Ass
4. \( \lnot \lnot R \) \( \checkmark \) \( \lnot \) Conc
5. \( P \) \( Q \lor \lnot R \) 1 \( \lor \) Elim
6. \( \lnot P \) \( \lnot R \)
   \( \times \) \( \times \)
   5, 6 \( \times \) 4, 6
7. \( Q \) \( \lnot R \)
   \( \times \)
   5 \( \lor \) Elim
8. \( \lnot Q \) \( \lnot R \)
   \( \times \) \( \times \)
   7, 8 \( \times \) 4, 8
9. \( 3 \rightarrow \) Elim
There are many proof systems.

1. \(((P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P)))\) by Ax2
2. \((P \to ((P \to P) \to P))\) by Ax1
3. \(((P \to (P \to P)) \to (P \to P))\) from 2, 1 by MP
4. \((P \to (P \to P))\) Ax1
5. \((P \to P)\) from 4, 3 by MP
Main types of proof system:

- Natural Deduction
- Trees (Tableaux)
- Axiomatic
- Sequent

The choice is largely a matter of taste.
Validity and Provability

Whatever method we use, we want it to have the following properties:

- **Soundness**: If a sentence is provable, then it is valid.
- **Completeness**: If a sentence is valid, then it is provable.
Soundness of K-trees
### Soundness of K-trees

We have many concepts of validity, and different trees rules for each.

<table>
<thead>
<tr>
<th>Validity Type</th>
<th>Rules</th>
</tr>
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<tbody>
<tr>
<td>K-valid</td>
<td>K-rules</td>
</tr>
<tr>
<td>T-valid</td>
<td>K-rules + Reflexivity</td>
</tr>
<tr>
<td>D-valid</td>
<td>K-rules + Seriality</td>
</tr>
<tr>
<td>K4-valid</td>
<td>K-rules + Transitivity</td>
</tr>
<tr>
<td>S4-valid</td>
<td>K-rules + Reflexivity + Transitivity</td>
</tr>
<tr>
<td>S4.2-valid</td>
<td>K-rules + Reflexivity + Transitivity + Convergence</td>
</tr>
<tr>
<td>S5-valid</td>
<td>S5-rules</td>
</tr>
</tbody>
</table>

...
Let’s show that the K-rules are sound for K-validity:

If a K-tree for a target sentence closes, then that sentence is K-valid.

How could we show this?

Let’s try a conditional proof:

• We assume there is a closed K-tree for some sentence $A$.
• We want to infer that $A$ is K-valid.
• We assume there is a closed K-tree for some sentence $A$.
• We want to infer that $A$ is K-valid. We want to infer that $A$ is true at all worlds in all Kripke models.

The guiding idea behind the tree method is to assume that the target sentence is false at some world ($w$) in some model, and derive a contradiction.

1. $\neg A \ (w)$

We are going to show that if a tree for $A$ closes, then the hypothesis that $A$ is false at some world in some Kripke model leads to contradiction.
Soundness of K-trees

• We assume there is a closed K-tree for some sentence $A$.
• We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
• We want to derive a contradiction.

1. $\neg A \ (w)$

The first node on the tree is a correct statement about $M$.
Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.

1. $\neg (B \rightarrow C)$ ($w$)
2. $B$ ($w$) (1)
3. $\neg C$ ($w$) (1)

After the first node is expanded, the new nodes are also correct statements about $M$. 
• We assume there is a closed K-tree for some sentence $A$.
• We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
• We want to derive a contradiction.

After node $i$ is expanded, the new node on at least one branch is also correct statement about $M$. 

\[
\begin{align*}
  i. & \quad B \lor C \quad (w) \\
  j. & \quad B \quad (w) \\
  k. & \quad C \quad (w)
\end{align*}
\]
Soundness of K-trees

- We assume there is a closed K-tree for some sentence $A$.
- We suppose that $A$ is false at some world $w$ in some Kripke model $M$.
- We want to derive a contradiction.

In general, we can show this:

If all nodes on some branch of a tree are correct statements about $M$, and the branch is extended by the K-rules, then all nodes on at least one of the resulting branches are still correct statements about $M$.

It follows that all nodes on some branch of the tree for $A$ are correct statements about $M$. 
Soundness of K-trees

• We assume there is a closed K-tree for some sentence \( A \).
• We suppose that \( A \) is false at some world \( w \) in some Kripke model \( M \).
• We want to derive a contradiction.
• The first node on the tree is a correct statement about \( M \).
• Whenever a node on the tree is expanded, all nodes on at least one branch are all correct statements about \( M \).
• But the tree is closed: every branch on the tree contains a contradictory pair

\[
\begin{align*}
\text{n.} & \quad B \quad (v) \\
\text{m.} & \quad \neg B \quad (v)
\end{align*}
\]

These two nodes can’t both be correct statements about \( M \).
Completeness of K-trees
Completeness of K-trees

We have shown

<table>
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Now we want to show

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<td>If a sentence is K-valid, then there is a closed K-tree for the sentence.</td>
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Completeness of K-trees

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We will prove something even stronger:

• If a sentence is K-valid, then any fully expanded K-tree for the sentence is closed.

Equivalently:

• If a fully expanded K-tree does not close, then the target sentence is not K-valid.
Completeness of K-trees

If a fully expanded K-tree does not close, then the target sentence is not K-valid.

- We assume that a fully expanded K-tree for a target sentence $A$ has an open branch.
- We want to infer that $A$ is false at some world in some model.

We already know how to construct such a model: we can read it off from any open branch!

All we need to show is that our method for reading off a model from open branches always provides a countermodel for the target sentence.
Suppose there is an open branch on a fully expanded tree.

Let $M$ be the model we read off from that branch.

We show that every node on the branch is a correct statement about $M$.

- The claim is obvious for sentence letters and negated sentence letters.
- Suppose $p \land q (w)$ is on the branch.
- Then $p (w)$ and $q (w)$ are on the branch.
- So $p$ is true at $w$ and $q$ at $w$ in $M$.
- So $p \land q$ is true at $w$ in $M$.
- And so on.
Completeness of K-trees

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

- We show that if there is a fully expanded but open K-tree for a sentence, then that sentence is not valid.
- We do this by showing that the model we can read off from an open branch on a fully expanded K-tree is always a countermodel for the target sentence.