# Logic 2: Modal Logic

## Lecture 8

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# Completeness of K-trees

# Completeness of K-trees

- A proof technique is **sound** if everything that's provable is valid.
- A proof technique is **complete** if everything that's valid is provable.

# Completeness of K-trees

We have shown

**Soundness**

If a K-tree for a target sentence closes, then that sentence is K-valid.

Now we want to show

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

# Completeness of K-trees

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

We will prove something even stronger:

- If a sentence is K-valid, then any fully expanded K-tree for the sentence is closed.

We prove the contraposition:

- If a fully expanded K-tree does not close, then the target sentence is not K-valid.

# Completeness of K-trees

If a fully expanded K-tree does not close, then the target sentence is not K-valid.

- We assume that a fully expanded K-tree for a target sentence $A$ has an open branch.
- We want to infer that $A$ is false at some world in some model.

We already know how to construct such a model: we can read it off from the open branch!

All we need to show is that our method for reading off a model from open branches always provides a countermodel for the target sentence.

# Completeness of K-trees

Suppose there is an open branch on a fully expanded tree.

Let $M$ be the model we read off from that branch.

We show that every node on the branch is a correct statement about $M$.

- The claim is obvious for sentence letters and negated sentence letters.
- Suppose $p \wedge q$ $(w)$ is on the branch.
- Then $p$ $(w)$ and $q$ $(w)$ are on the branch.
- So $p$ is true at $w$ and $q$ at $w$ in $M$.
- So $p \wedge q$ is true at $w$ in $M$.
- And so on.

# Completeness of K-trees

**Completeness**

If a sentence is K-valid, then there is a closed K-tree for the sentence.

- We show that if there is a fully expanded but open K-tree for a sentence, then that sentence is not valid.
- We do this by showing that the model we can read off from an open branch on a fully expanded K-tree is always a countermodel for the target sentence.

# Axiomatic proofs

# Axiomatic proofs

An **axiomatic proof** is a list of sentences each of which is either

- an axiom of the proof system, or
- follows from earlier sentences by a rule of the proof system.

# Axiomatic proofs

**An axiomatic calculus for classical propositional logic:**

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**MP**) If $A$ and $A \to B$ occur on a proof, you may append $B$.

# Axiomatic proofs

**An axiomatic calculus for the modal logic K:**

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**MP**) If $A$ and $A \to B$ occur on a proof, you may append $B$.

(**Nec**) If $A$ occurs on a proof, you may append $\Box A$.

# Axiomatic proofs

The axiomatic method is useful to summarize a logical system.

Example: Which $\mathfrak{L}_M$-sentences are valid in the class of reflexive Kripke models?

1. The ones that can be proved with these 18 tree rules...
2. The ones that can be proved with these 26 natural deduction rules...
3. All propositional tautologies,
   all instances of $\Box(A \to B) \to (\Box A \to \Box B)$,
   all instances of $\Box A \to A$,
   and anything that can be derived from these by Modus Ponens and Necessitation.

# Axiomatic proofs

Sometimes, the modal operators have no intuitive interpretation as “true at all accessible worlds”, but their logic nonetheless matches validity in some class of Kripke frames.

Example: $\Box A \iff A$ is true in virtue of its form.

- $\Box(A \to B) \to (\Box A \to \Box B)$ is plausibly valid.
- $\Box A \to A$ is plausibly valid.
- $\Diamond A \to \Box\Diamond A$ is plausibly valid.
- If $A$ is valid, then so is $\Box A$.
- If $A$ and $A \to B$ are valid, then so is $B$.

# Axiomatic proofs

- A proof technique is **sound** if everything that's provable is valid.
- A proof technique is **complete** if everything that's valid is provable.

Let's prove soundness and completeness for the axiomatic calculus for K.

# Soundness

# Soundness

We want to show that **if**

- there is a derivation of a sentence $A$ from **A1–A3**, **K** by **MP** and **Nec**

**then**

- $A$ is true at all worlds in all Kripke models.

# Soundness

We show that

1. Every instance of **A1**, **A2**, **A3**, and **K** is K-valid.
2. If **MP** and **Nec** are applied to K-valid sentences, then the newly added sentence is also K-valid.

# Soundness

1. Every instance of $A \rightarrow (B \rightarrow A)$ is true at every world in every Kripke model.
2. Every instance of $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ is true at every world in every Kripke model.
3. Every instance of $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$ is true at every world in every Kripke model.
4. Every instance of $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ is true at every world in every Kripke model.

## Soundness

1. If $A \rightarrow B$ is true at every world in every Kripke model, and so is $A$, then $B$ is also true at every world in every Kripke model.
2. If $A$ is true at every world in every Kripke model, then $\Box A$ is true at every world in every Kripke model.

# Completeness

# Completeness

**To show:**

If a sentence is K-valid, then it is provable from **A1–A3** and **K** by **MP** and **Nec**.

For short: If $A$ is K-valid, then $A$ is K-provable.

The argument will be by contraposition:

We'll show that if $A$ is not K-provable, then $A$ is not K-valid.

# Completeness

To show: If $A$ is not K-provable, then $A$ is not K-valid.

- We assume that $A$ is not K-provable.
- We give a countermodel to show that $A$ is not K-valid.
- We use the same countermodel for every sentence: the **canonical model for K.**

# Completeness

To show: If $A$ is not K-provable, then $A$ is false at some world in the canonical model for K.

Canonical models are defined so that

- The worlds are sets of sentences.
- A sentence is true at a world iff it is a member of the world.
- Whenever a sentence is not provable, its negation is a member of some world.

## Completeness

- A sentence is true at a world iff it is a member of the world.

What will the canonical model for K look like?

The set of sentences true at any world $w$ in any Kripke model $M$ is

- **maximal**: For every sentence $B$, the set contains either $B$ or $\neg B$;
- **K-consistent**: There is no sentence $B$ true at $w$ for which $\neg B$ is K-provable.

In any Kripke model $M$,

- $wRv$ only if $M, v \models A$ for all sentences $A$ for which $M, w \models \Box A$.

# Completeness

The **canonical model** $M_K$ for K is the Kripke model $(W, R, V)$, where

- $W$ is the set of all maximal K-consistent sets of $\mathfrak{L}_M$-sentences.
- $wRv$ iff $v$ contains every sentence $A$ for which $w$ contains $\Box A$.
- For every sentence letter $\rho$ and world $w$, $V(\rho, w) = 1$ iff $\rho \in w$.

**Canonical Model Lemma**

$M_K, w \models A$ iff $A \in w$ (for any sentence $A$).

# Completeness

**To show:** If $A$ is K-valid then $A$ is K-provable.

**We show:** If $A$ is not K-provable then $A$ is false at some world in the canonical model for K.

1. Assume $A$ is not K-provable.
2. Then $\neg A$ is a member of some maximal K-consistent set. (Lindenbaum's Lemma)
3. So $\neg A \in w$ for some world $w$ in $M_{K}$.
4. So $M_{K}, w \models \neg A$, by the Canonical Model Lemma.
5. So $M_{K}, w \not\models A$.

# More Completeness Proofs

# More Completeness Proofs

**Completeness of S4:**

> *If a sentence is S4-valid, then it is provable from **A1–A3**, **K**, **T**, and **4** by **MP** and **Nec**.*

S4-valid means true at all worlds in all reflexive and transitive Kripke models.

**We show:**

> *If a sentence is not S4-provable then it is not S4-valid.*
>
> *If a sentence is not S4-provable then it is false at some world in the canonical model for S4, and this model is reflexive and transitive.*

## Canonical Model

The **canonical model** $M_{S4}$ for S4 is the Kripke model $\langle W, R, V \rangle$, where

- $W$ is the set of all maximal S4-consistent sets of $\mathfrak{L}_M$-sentences.
- $wRv$ iff $v$ contains every sentence $A$ for which $w$ contains $\Box A$.
- For every sentence letter $p$ and world $w$, $V(p, w) = 1$ iff $p \in w$.

## Canonical Model Lemma

$M_{S4}, w \models A$ iff $A \in w$

# More Completeness Proofs

**To show:** If $A$ is S4-valid then $A$ is provable in the axiomatic calculus for S4.

**We show:** If $A$ is not S4-provable then $A$ is false at some world in the canonical model for S4.

1. Assume $A$ is not S4-provable.
2. Then $\neg A$ is a member of some maximal S4-consistent set. (Lindenbaum's Lemma)
3. So $\neg A \in w$ for some world $w$ in $M_{S4}$.
4. So $M_{S4}, w \models \neg A$, by the Canonical Model Lemma.
5. So $M_{S4}, w \not\models A$.

Still need to show that $M_{S4}$ is reflexive and transitive!

# Strong completeness

# Strong completeness

Claims about entailment can usually be converted into claims about validity:

$A_1, \ldots, A_n \models B$ iff $\models (A_1 \wedge \ldots \wedge A_n) \rightarrow B$

But not if there are infinitely many premises.

$A_1, A_2, A_3, \ldots \models B$

Proofs are finite.

So any derivation of $B$ from $A_1, A_2, A_3, \ldots$ can only make use of finitely many of the premises.

Let's say that $B$ is derivable from $A_1, A_2, A_3, \ldots$ (in a given proof system) if there are finitely many sentences $A_1, \ldots, A_n$ from $A_1, A_2, A_3, \ldots$ such that $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$ is provable (in the system).

# Strong completeness

**Strong Completeness**

If $A_1, A_2, A_3, \ldots \models B$ then $B$ is derivable from $A_1, A_2, A_3, \ldots$.

**Proof outline:**

1. Assume $B$ is not K-derivable from $A_1, A_2, A_3, \ldots$.
2. Then there are no $A_1, \ldots, A_n$ from $A_1, A_2, A_3, \ldots$ such that $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$ is K-provable.
3. Then $A_1, A_2, A_3, \ldots \cup \{\neg B\}$ is included in some maximal K-consistent set. (Lindenbaum's Lemma)
4. So $A_1, A_2, A_3, \ldots \cup \{\neg B\}$ is included in some world $w$ in $M_{\text{K}}$.
5. So by the Canonical Model Lemma, $M_{\text{K}}, w \models_K A_i$ for all $A_i$, and $M_{\text{K}}, w \not\models_K B$.
6. So $A_1, A_2, A_3, \ldots \not\models_K B$.

# Strong completeness

## Strong Completeness

If $A_1, A_2, A_3, \ldots \models B$ then $B$ is derivable from $A_1, A_2, A_3, \ldots$.

Corollary:

## Compactness

Whenever a conclusion is entailed by infinitely many sentences then it is entailed by a finite subset of those sentences.