Logic 2: Modal Logic

Lecture 10

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Multi-Modal Logic

Multi-Modal Logic

If we want to talk about several agents, we need a multi-modal logic.

Definition

A **multi-modal Kripke model** consists of

- a non-empty set W,
- a finite set of binary relation $R_1, R_2, \ldots, R_n$ on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ and each element of W a truth-value.

In epistemic logic, v is R_i-accessible from w iff v is compatible with the information agent i has at world w.

Multi-Modal Logic

The **language of multi-modal propositional logic** has several boxes $\Box_1, \Box_2, \ldots, \Box_n$ and diamonds $\Diamond_1, \Diamond_2, \ldots \Diamond_n$.

$$M, w \models \Box_i A \quad \text{iff } M, v \models A \text{ for all } v \text{ such that } wR_i v.$$
$$M, w \models \Diamond_i A \quad \text{iff } M, v \models A \text{ for some } v \text{ such that } wR_i v.$$

Multi-Modal Logic

As before, we write the boxes as ‘K’ and the diamonds as ‘M’.

- $\mathrm{M}_1\, p$
- $\mathrm{K}_1\, p$
- $\mathrm{K}_1\, \mathrm{M}_2\, p$
- $\mathrm{K}_1\, p \rightarrow \mathrm{M}_2\, p$
- $\mathrm{K}_1\, p \rightarrow \mathrm{K}_2\, \mathrm{K}_1\, p$

Muddy Children

Muddy Children

Muddy Children

Three children have been playing outside.

Mother: “At least one of you has mud on their face. Do you know if you have mud on your face?”

All children: “No.”

Mother: “Do you know if you have mud on your face?”

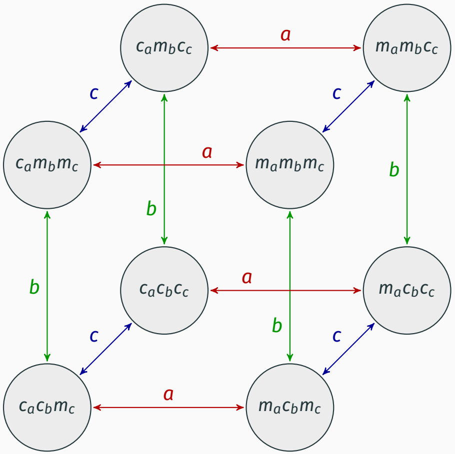
Two children: “Yes”.

Mother: “Do you know if you have mud on your face?”

All children: “Yes”.

Muddy Children

Mother: “At least one of you has mud on their face.”

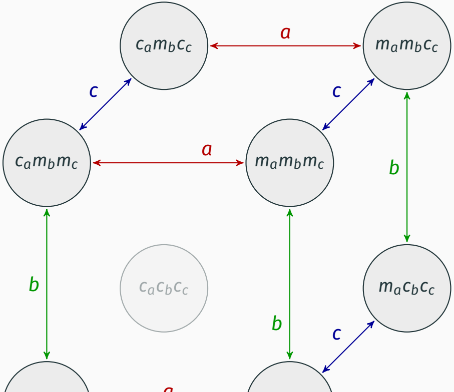


Muddy Children

Mother: “At least one of you has mud on their face.”

Mother: “Do you know if you have mud on your face?”

All children: “No.”

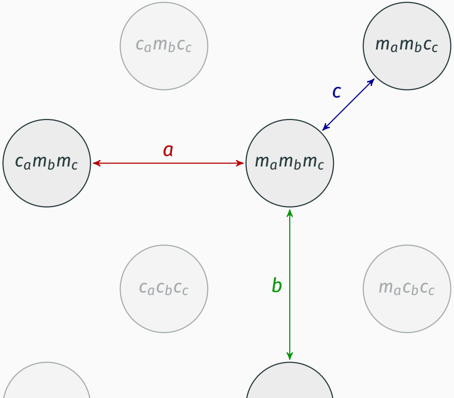


Muddy Children

Mother: “Do you know if you have mud on your face?”

All children: “No.” Mother: “Do you know if you have mud on your face?”

Two children: “Yes.”



Muddy Children

Mother: “Do you know if you have mud on your face?”

Two children: “Yes.” Mother: “Do you know if you have mud on your face?”

All children: “Yes”.

Interaction principles

Interaction principles

In multi-modal logics, we can impose constraints on individual accessibility relations:

- R_1 is reflexive
- R_2 is transitive
- etc.

but also on how different relations interact:

- if wR_1v then wR_2v
- if wR_1v then vR_2w
- if wR_1v and vR_2u then wR_3u
- etc.

Interaction principles

Constraints on the interaction between accessibility relations correspond to **interaction schemas** that link different operators.

$\Diamond_1 A \rightarrow \Diamond_2 A$

$\Diamond_1 A \rightarrow \Box_2 \Diamond_1 A$

etc.

Interaction principles

An interaction principle for multi-agent knowledge:

$$K_1 K_2 A \rightarrow K_1 A$$

But this follows from the **T**-schema for K_2:

1.	$K_2 A \rightarrow A$	(T)
2.	$K_1(K_2 A \rightarrow A)$	(1, Nec)
3.	$K_1(K_2 A \rightarrow A) \rightarrow (K_1 K_2 A \rightarrow K_1)$	(K)
4.	$K_1 K_2 A \rightarrow K_1 A$	(2, 3, MP)

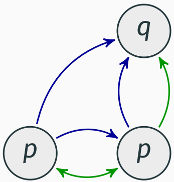
Mutual Knowledge

Mutual Knowledge

Let $\mathrm{E}_G A$ mean that every member of group G knows A.

$$M, w \models \mathrm{E}_G A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_G v$$

How is R_G related to $R_1, R_2, \ldots, R_n$? $wR_G v$ iff $wR_1 v$ or $wR_2 v$ or ...or $wR_3 v$



Mutual Knowledge

Let $\mathrm{E}_G A$ mean that every member of group G knows A.

$$M, w \models \mathrm{E}_G A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_G v$$

Interaction Principle:

$$\mathrm{E}_G A \longleftrightarrow \mathrm{K}_1 A \wedge \mathrm{K}_2 A \wedge \ldots \wedge \mathrm{K}_n A$$

Common Knowledge

Common Knowledge

Let $\mathsf{C}_G A$ mean that every member of group G knows A, everyone knows that everyone knows A, everyone knows that everyone knows A, and so on.

$$M, w \models \mathsf{C}_G A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_C v$$

How is R_C related to $R_1, R_2, \ldots, R_n$?

$wR_C v$ iff there is a path from w to v following R_1, R_2, ..., R_3.

Common Knowledge

Let $\mathsf{C}_G A$ mean that every member of group G knows A, everyone knows that everyone knows A, everyone knows that everyone knows A, and so on.

$$M, w \models \mathsf{C}_G A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_C v$$

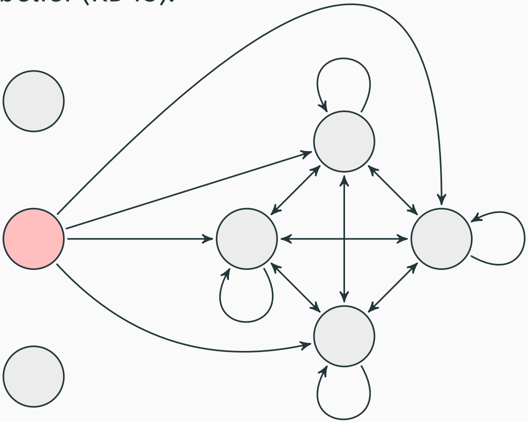
Interaction Principles:

$$\mathsf{C}_G A \leftrightarrow (A \wedge \mathsf{E}_G \mathsf{C}_G A)$$

$$(A \wedge \mathsf{C}_G(A \rightarrow \mathsf{E}_G)) \rightarrow \mathsf{C}_G A$$

Knowledge and Belief

A simple picture of belief (KD45):



Knowledge and Belief

Let $\mathsf{B}A$ mean that some (fixed) agent believes A.

$$M, w \models \mathsf{K}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_K v$$

$$M, w \models \mathsf{B}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_B v$$

A plausible interaction principle: $\mathsf{K}A \rightarrow \mathsf{B}A$

What does this mean for R_B and R_K?

$$\mathsf{K}A \rightarrow \mathsf{B}A$$

$$\neg \mathsf{B}A \rightarrow \neg \mathsf{K}A$$

$$\Diamond_B A \rightarrow \Diamond_K A$$

Whenever $wR_B v$ then $wR_K v$.

Candidate Interaction Principles for B and K:

$$(\textbf{KB}) \quad \mathsf{K}A \rightarrow \mathsf{B}A$$

$$(\textbf{PI}) \quad \mathsf{B}A \rightarrow \mathsf{K}\mathsf{B}A$$

$$(\textbf{NI}) \quad \neg\mathsf{B}A \rightarrow \mathsf{K}\neg\mathsf{B}A$$

$$(\textbf{SB}) \quad \mathsf{B}A \rightarrow \mathsf{B}\mathsf{K}A$$

These entail

$$(\textbf{B4}) \quad \mathsf{B}A \rightarrow \mathsf{B}\mathsf{B}A$$

$$(\textbf{B5}) \quad \neg\mathsf{B}A \rightarrow \mathsf{B}\neg\mathsf{B}A$$

$$(\textbf{KG}) \quad \mathsf{M}\mathsf{K}A \rightarrow \mathsf{K}\mathsf{M}A$$

Knowledge and Possibility

Knowledge and Possibility

Let $\Diamond A$ mean that A is possible, in some circumstantial sense.

$$M, w \models \mathsf{K}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_K v$$

$$M, w \models \Diamond A \text{ iff } M, v \models A \text{ for some } v \text{ such that } wR_C v$$

The verificationist **principle of knowability**: $A \to \Diamond \mathsf{K} A$

1. Let p be any unknown truth. So $p \wedge \neg \mathsf{K} p$.
2. By the knowability principle, $\Diamond \mathsf{K}(p \wedge \neg \mathsf{K} p)$.
3. $\mathsf{K}(p \wedge \neg \mathsf{K} p)$ entails $\mathsf{K} p \wedge \mathsf{K} \neg \mathsf{K} p$.
4. $\mathsf{K} \neg \mathsf{K} p$ entails $\neg \mathsf{K} p$.
5. So $\mathsf{K}(p \wedge \neg \mathsf{K} p)$ entails both $\mathsf{K} p$ and $\neg \mathsf{K} p$.
6. So $\neg \Diamond \mathsf{K}(p \wedge \neg \mathsf{K} p)$.