Logic 2: Modal Logic

Lecture 21

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Intermediate Logic

What we've covered:

- Translating from English into a formal language
- Reasoning in a formal language
- Different styles of proof
- The concept of a model
- Soundness and completeness (and compactness)
- Three-valued logic, supervaluation, 2D semantics, ...
- Sets, relations, orders, ...

What we've covered:

- Epistemic logic
- Deontic logic
- Temporal logic
- Conditionals

Review: Language

Review: Language

Modal logic is widely used to formalise reasoning with **non-truth-functional** concepts.

- knowledge
- belief
- provability
- obligation and permission
- what will or was the case
- what could have been the case
- what would have been the case if so-and-so had been the case
- ...

(What is the logic of ‘it is true that’?)

Review: Language

To reason with these concepts, we add **new sentence operators** to the language of classical propositional or predicate logic.

- $\Box$ – can mean anything, often: it is necessity that
- K – it is known that
- B – it is believed that
- O – it is obligatory that
- G – it is always going to be the case that
- H – it has always been the case that
- $A \Box\!\!\rightarrow B$ – if A had been the case then B would have been the case
- ...

Review: Language

Heuristics for **translating from English into the language of modal logic**:

- First paraphrase the original English sentence in such a way that all relevant modal elements are turned into sentence operators: ‘it is necessary that’, ‘it is possible that’, ‘it is required that’.
  Avoid ‘if–then’ constructions in your paraphrase.
- Make sure your sentence letters stand for complete sentences that don’t contain any relevant logical expressions.
- Check if you can think of a scenario in which your translation and the original sentence have different truth-values. Try edge cases!
- Avoid $A \rightarrow \Box B$.
- Avoid $\Diamond(A \rightarrow B)$.

(1.a) We may have to pay a fine.

Paraphrase:

- It is (epistemically) possible that it is required that we pay a fine.

Translation:

- $\Diamond \mathrm{O}\, p$
  p: We pay a fine.

Review: Language

(1.b) If the sky is clear, it can't be raining.

Paraphrase:

- If the sky is clear, then it is not (epistemically) possible that it is raining.

Translation:

- $p \rightarrow \neg\Diamond q$?
  p: the sky is clear
  q: it is raining

It is easy to think of scenarios in which (1.b) is true, and yet p is true and $\neg\Diamond q$ is false.

Review: Language

(1.b) If the sky is clear, it can't be raining.

Paraphrase:

- It is certain that if the sky is clear, then it is not raining.
- It is not possible that the sky is clear and it is raining.

Translation:

- $\Box(p \rightarrow \neg q)$
  p: the sky is clear
  q: it is raining
- $\neg\Diamond(p \wedge q)$

Review: Language

(1.c) The dog must not be left alone in the home for longer than 5 hours or overnight.

Paraphrase:

- It is not allowed that the dog is left alone in the home for longer than 5 hours or the dog is left alone in the home overnight.

Translation:

- $\neg \mathrm{P}(p \vee q)$
  p: The dog is left alone in the home for longer than 5 hours.
  q: The dog is left alone in the home overnight.
- $\mathrm{O} \neg (p \vee q)$
- $\mathrm{O} \neg p \wedge \mathrm{O} \neg q$

Review: Language

(1.d) A nuclear strike might lead to many innocent deaths and so can't be the right thing to do.

- $\Diamond p \wedge \neg\Diamond \mathrm{O}\, q$
  p: A nuclear strike leads to many innocent deaths.
  q: A nuclear strike is carried out

Wrong:

- $\Box p \rightarrow \mathrm{O}\,\neg q$

Review: Language

Exam question 1:

Translate the following sentences, as well as possible, into a suitable modal language. (The resources of modal predicate logic are only needed for d.)

(a) You can keep your shoes on.

(b) ...

(c) ...

(d) ...

(4.b) Give an argument against the hypothesis that an indicative conditional ‘if *A* then *B*’ is true iff *A* and *B* are both true.

Good answers:

1. ‘If *A* then *B*’ does not entail ‘if *B* then *A*’.
2. ‘If Oswald didn’t kill Kennedy then someone else did’ is true, but the corresponding conjunction is false.

Less good:

3. ‘If *A* then *B*’ is true whenever *A* is false.
4. ‘If *A* then *B*’ is true only if there is an explanatory connection between *A* and *B*.

Review: Models

Review: Models

Informally, A logically entails B (for short, $A \models B$) iff there is no conceivable scenario in which A is true and B is false, under any interpretation of the non-logical vocabulary.

We represent a conceivable scenario and an interpretation of the non-logical vocabulary by a **model**.

A entails B iff $A \to B$ is valid – true in every scenario under every interpretation.

Review: Models

To clarify the logic of modal operators, we often use **possible-worlds models**.

Guiding intuition:

- *p* is possible iff *p* is true at some (relevantly) possible world.
- *p* is necessary iff *p* is true at every (relevantly) possible world.

Review: Models

A **Kripke model** for modal propositional logic consists of

- a set of "worlds" W,
- a binary "accessibility" relation R on W, and
- an interpretation function V that assigns to each sentence letter of $\mathfrak{L}_M$ a subset of W.

Kripke semantics for modal propositional logic specifies, for any world w in any Kripke model M, and any $\mathfrak{L}_M$-sentence A, whether A is true at w in M.

Review: Models

The **accessibility relation** in Kripke model represents different things, depending on the application.

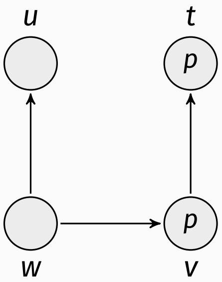
- wRv iff v is compatible with the laws of physics at w.
- wRv iff v is compatible with the knowledge at w.
- wRv iff v is compatible with the norms at w.
- wRv iff v is compatible with the essence of things.
- wRv iff v is later than w.
- ...

Review: Models

Possible exam question:

In the Kripke model on the right, p is true at v and t and false at w and u.

1. At which worlds in the model is $\Box p$ true?
2. For each world in the model, find an $\mathfrak{L}_M$-sentence that is true only at that world.



Review: Models

Different interpretations of accessibility imply different formal constraints on Kripke models.

- R is **reflexive** if every world is accessible from itself.
- R is **serial** if every world can access some world.
- R is **universal** if every world can access every world.
- R is **transitive** if whenever wRv and vRu then wRu.
- R is **euclidean** if whenever wRv and wRu then vRu.
- ...

Review: Models

By imposing constraints on Kripke models, we get different **logics** (or **systems**).

- A sentence is **K-valid** if it is true at all worlds in all Kripke models.
- A sentence is **T-valid** if it is true at all worlds in all Kripke models in which R is reflexive.
- A sentence is **D-valid** if it is true at all worlds in all Kripke models in which R is serial.
- A sentence is **S5-valid** if it is true at all worlds in all Kripke models in which R is universal (or: an equivalence relation)
- A sentence is **S4-valid** if it is true at all worlds in all Kripke models in which R is reflexive and transitive.
- ...

There are infinitely many modal logics.

Review: Models

The **system K** is the set of all K-valid sentences.

The **system T** is the set of all T-valid sentences.

...

Review: Models

- A formal sentence can be true at a world in a model. It can't be valid at a world.
- A sentence can be valid in a class of Kripke models. It can't be true in such a class.
- A system of modal logic does not have an accessibility relation.
- A system of modal logic does not have any rules.
- A system of modal logic does not contain any schemas.

(4.c) Can you explain why S5 is not plausible as the logic of belief, where the box is read as (e.g.) 'Alice believes that'?

Most obvious answer:

In S5, $\Box A$ entails A. But the fact that Alice believes that it is raining doesn't entail that it is raining.

Review: Models

Possible exam questions:

1. Give an example of an $\mathfrak{L}_M$-sentences that is K-valid but that is not an instance of the (K)-schema $\Box(A \to B) \to (\Box A \to \Box B)$.
2. Explain why every K-valid $\mathfrak{L}_M$-sentence is S4-valid.

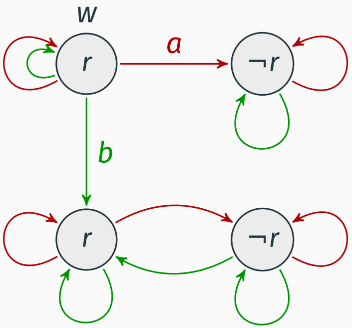
Review: Models

(3.a) Draw a Kripke model for epistemic logic, in which the following is true at *w*: Alice does not know whether it is raining. (Let *r* translate 'it is raining'.)

- A model only directly fixes the truth-value of atomic sentences at worlds.
- Epistemic models are reflexive.

Review: Models

(3.b) Draw a Kripke model for multi-modal epistemic logic in which the following are true at *w*: Alice does not know whether it is raining; Alice knows that Bob knows whether it is raining; but Bob does not know that Alice knows that Bob knows whether it is raining. (Let *r* translate 'it is raining'.)



Review: Models

(4.e) Is $\Box(\Box A \to A) \to (\Diamond\Box A \to \Box A)$ valid in the class of Kripke models in which R is a linear order? If yes, explain briefly. If no, give a counterexample.

For a counterexample, $\Box(\Box A \to A)$ and $\Diamond\Box A$ must be true while $\Box A$ is false.

schema	true at a world w in a linear model iff
$\Box A$	A is true at all points "in the future" of w.
$\Box A \to A$	Either A is false somewhere in the future or it is true at w.
$\Box(\Box A \to A)$	A is true at any future point after which A is always true.
$\Box(\Box A \to A)$	No $\neg A$-point in the future is followed only by A-points.
$\Diamond\Box A$	There is a point in the future after which A is always true.

Review: Models

For a counterexample, $\Box(\Box A \to A)$ and $\Diamond\Box A$ must be true while $\Box A$ is false.

$\Box(\Box A \to A)$	No $\neg A$-point in the future is followed only by A-points.
$\Diamond\Box A$	There is a point in the future after which A is always true.
$\Box A$	A is true at all points "in the future" of w.

Example:

$W =$ the set of real numbers
wRv iff $w < v$
$V(p) = \{x : x \geq 1\}$

Here, $\Box(\Box p \to p)$ and $\Diamond\Box p$ are true at 0, and $\Box p$ is false.

Review: Models

NB: $\Box(\Box A \rightarrow A)$ is valid on all and only the frames that are shift reflexive. But an instance of $\Box(\Box A \rightarrow A)$ can be true at a world in a model even if the model isn't shift reflexive.