Logic 2: Modal Logic

Lecture 22

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Review: Models

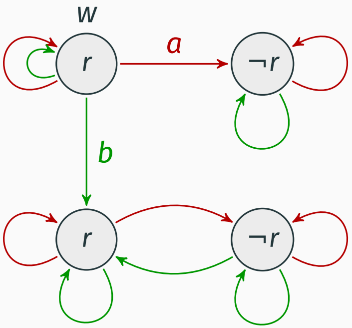
Review: Models

(3.a) Draw a Kripke model for epistemic logic, in which the following is true at *w*: Alice does not know whether it is raining. (Let *r* translate 'it is raining'.)

- The interpretation function only fixes the truth-value of non-modal atomic sentences at each world.
- Epistemic models are reflexive.

Review: Models

(3.b) Draw a Kripke model for multi-modal epistemic logic in which the following are true at *w*: Alice does not know whether it is raining; Alice knows that Bob knows whether it is raining; but Bob does not know that Alice knows that Bob knows whether it is raining. (Let *r* translate 'it is raining'.)



(4.e) Is $\Box(\Box A \to A) \to (\Diamond\Box A \to \Box A)$ valid in the class of Kripke models in which R is a linear order? If yes, explain briefly. If no, give a counterexample.

For a counterexample, $\Box(\Box A \to A)$ and $\Diamond\Box A$ must be true while $\Box A$ is false.

$\Box(\Box A \to A)$	No $\neg A$-point in the future is followed only by A-points.
$\Diamond\Box A$	There is a point in the future after which A is always true.
$\Box A$	A is true at all points "in the future" of w.

$W =$ the set of real numbers
wRv iff $w < v$
$V(p) = \{x : x \geq 1\}$

Here, $\Box(\Box p \to p)$ and $\Diamond\Box p$ are true at 0, and $\Box p$ is false.

Review: Models

NB: $\Box(\Box A \rightarrow A)$ corresponds to shift reflexivity. But an instance of $\Box(\Box A \rightarrow A)$ can be true at a world in a model even if the model isn't shift reflexive.

Review: Frames and correspondence

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A Kripke model has three parts: W, R, V.

When we define validity (or entailment) in terms of a class of Kripke models, we never put constraints on V.

- A is **T**-valid iff A is valid in the class of Kripke models with a reflexive accessibility relation.
- A is **K′**-valid iff A is valid in the class of Kripke models with finitely many worlds.
- A is **X**-valid iff A is valid in the class of Kripke models in which $V(p) = W$. ???

p is X-valid, but q is not!

Review: Frames and correspondence

A Kripke model has three parts: W, R, V.

When we define validity (or entailment) in terms of a class of Kripke models, we never put constraints on V.

We effectively define validity with respect to a class of Kripke frames.

A **frame** is a model without an interpretation function.

A sentence is **valid on a frame** iff it is true at all worlds in all models based on that frame.

Review: Frames and correspondence

Constraints on the accessibility relation often **correspond** to modal schemas, in the sense that all instances of the schema are valid on a frame iff the frame satisfies the constraint.

	Schema	*Corresponding Frame Condition*
T	$\Box A \to A$	R is reflexive: every world is accessible from itself
D	$\Box A \to \Diamond A$	R is serial: every world can access some world
B	$A \to \Box\Diamond A$	R is symmetric: whenever wRv then vRw
4	$\Box A \to \Box\Box A$	R is transitive: whenever wRv and vRu, then wRu
5	$\Diamond A \to \Box\Diamond A$	R is euclidean: whenever wRv and wRu, then vRu
		such that vRt and uRt

Review: Frames and correspondence

Sentences or schemas that contain more than one type of box or diamond are called **interaction principles**.

Interaction principles often correspond to joint constraints on different accessibility relations.

Review: Frames and correspondence

(4.f) State a condition on multi-modal epistemic frames that corresponds to $\neg K_1 A \rightarrow K_2 \neg K_1 A$, and outline a proof of the correspondence.

Condition: If wR_1v and wR_2u then uR_1v.

If $\neg K_1 A \rightarrow K_2 \neg K_1 A$ is false at a world w then $\neg K_1 A$ is true at w and $K_2 \neg K_1 A$ is false.

This means that there are worlds v and u such that wR_1v and $\neg A$ at v, and wR_2u and $K_1 A$ at u. This is impossible if uR_1v.

I.e., if the Condition holds then all instances of the schema are valid.

For the other direction, suppose the Condition does not hold. So there are worlds with wR_1v, wR_2u but not uR_1v. Let V make p false at v and true at all other worlds. Then $\neg K_1 p$ at w, but $K_1 p$ at u and so $\neg K_2 \neg K_1 p$ at w.

Review: Proofs

Review: Proofs

For almost every conception of validity (or logics) that we've studied, there are proof methods that allow us to demonstrate that a sentence is valid.

A method is **sound** if anything that's provable with the method is valid.

A method is **complete** if anything that's valid is provable with the method.

Review: Proofs

In a **tree proof** for a sentence A, we start with a node $\neg A\,(w)$.

Then we expand the nodes on the tree in accordance with the tree rules for the relevant system.

If the tree closes, the target sentence A is valid.

An **axiomatic proof** for a sentence A is a list of sentences each of which is either an axiom (for the relevant system) or follows from earlier items by one of the rules (for the system).

Review: Proofs

(4.h) Give a proof of $\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{H}\,p$ in the axiomatic calculus for K_t4.

1.	$\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{F}\,\mathsf{H}\,p$	(Con2)
2.	$\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{F}\,\mathsf{F}\,\mathsf{H}\,p$	(Con2)
3.	$\mathsf{F}\,\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{F}\,\mathsf{H}\,p$	(G4, Dual)
4.	$\mathsf{H}\,\mathsf{F}\,\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{F}\,\mathsf{H}\,p$	(3, KH, Nec)
5.	$\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{F}\,\mathsf{H}\,p$	(2, 5)
6.	$\mathsf{F}\,\mathsf{H}\,p \rightarrow p$	(Con1, Dual)
7.	$\mathsf{H}\,\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{H}\,p$	(6, KH, Nec)
8.	$\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{H}\,p$	(5, 7)
9.	$\mathsf{H}\,\mathsf{F}\,\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{H}\,p$	(8, KH, Nec)
10.	$\mathsf{H}\,p \rightarrow \mathsf{H}\,\mathsf{H}\,p$	(1, 9)

Review: Extra parameters

Review: Extra parameters

In Ockhamist semantics, truth is defined relative to an extra parameter that does not reflect an aspect of a conceivable scenario: a history.

We also need an extra (time or world) parameter if we want a 'Now' operator (or the modal analogue, 'Actually').

In first-order logic, we have an extra parameter g for assignment functions.

The method of supervaluation can be used to reduce extra parameters.

$M, t \models A$ iff $M, t, h \models A$ for all histories h through t.

Review: Ordering models

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Operators for obligation and permission arguably quantify over the best of the circumstantially accessible worlds.

To make this explicit, models of deontic logic need

- a circumstantial accessibility relation, and
- an ordering of worlds as better and worse.

Review: Ordering models

With deontic ordering models, we can also define a conditional obligation operator:

$O(p/q)$ is true at w iff p is true at the best of the circumstantially accessible worlds at which q is true.

Formally, $A \Box\!\!\rightarrow B$ has the same interpretation. The ordering is interpreted not as “better relative to w” but as “more similar to w”.

4.g) Suppose for every world there is an even better world, relative to some system of norms. Give a (plausible) new semantics for O in deontic ordering models so that the (D)-schema remains valid. You can assume that the circumstantial accessibility relation is serial.

Review: Ordering models

The old semantics:

$\mathrm{O}A$ is true at w iff A is true at all the best of the circumstantially accessible worlds.

Problem: If among the (circumstantially) accessible worlds, for every world there is an even better world, then there are no best of the accessible worlds.

Then $M, w \models \mathrm{O}A$ for all A, and $M, w \models \mathrm{P}A$ for no A.

So the **D**-schema $\mathrm{O}A \rightarrow \mathrm{P}A$ fails.

Review: Ordering models

New semantics:

$\mathrm{O}\,A$ is true at w iff for every circumstantially accessible world, as you move towards better worlds, A eventually becomes true and remains true.

Formally:

$M, w \models \mathrm{O}\,A$ iff for all u with wRu there is some v with wRv such that (i) not $u \prec_w v$ and (ii) for all t with wRt, if not $v \prec_w t$ then $M, t \models A$.

If R is serial, $\mathrm{O}\,A \to \mathrm{P}\,A$ is now valid.

Review: Conditionals

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Some think subjunctive conditionals in English should be formalized in terms of $\Box\!\!\rightarrow$.

Others think they are strict conditionals $A \strictif B$.

Some think indicative conditionals in English should be formalized in terms of $\Box\!\!\rightarrow$.

Others think they are strict conditionals $A \strictif B$.

Others think they are material conditionals $A \rightarrow B$.

Others think indicative conditionals do not express propositions at all.

Review: Modal Predicate Logic

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Modal predicate logic is a multi-modal logic in which quantifiers of the form $\forall x$ and $\exists x$ range over different things as quantifiers of the form $\Box$ and $\Diamond$.

Two main questions arise in the semantics of modal predicate logic:

- Can the domain of individuals vary from world from world?
- Is the reference of names and variables constant from world to world?

Logic-related courses in Y4

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- Logic, Computability, and Incompleteness (Schweizer)
- Puzzles and Paradoxes (Rabern)
- Belief, Desire, and Rational Choice (Schwarz)