Logic 2: Modal Logic

Lecture 14

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Review

Review

Temporal operators:

- F: It will (at some point) be the case that
- G: It is always going to be the case that
- P: It was (at some point) the case that
- H: It has always been the case that

Review

Temporal Model

A **temporal model** consists of

- a non-empty set T (of “times”),
- a binary relation $<$ on T (the **precedence relation**),
- a function V that assigns to each sentence letter of $\mathfrak{L}_T$ and each member of T a truth-value (1 or 0).

Review

Standard Temporal Semantics

(g) $M, t \models \mathrm{F}A$ iff $M, s \models A$ for some s such that $t < s$.

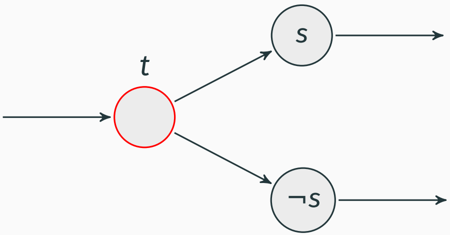
(h) $M, t \models \mathrm{G}A$ iff $M, s \models A$ for all s such that $t < s$.

(i) $M, t \models \mathrm{P}A$ iff $M, s \models A$ for some s such that $s < t$.

(j) $M, t \models \mathrm{H}A$ iff $M, s \models A$ for all s such that $s < t$.

Branching time

Branching time



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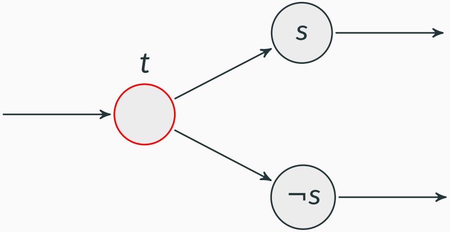
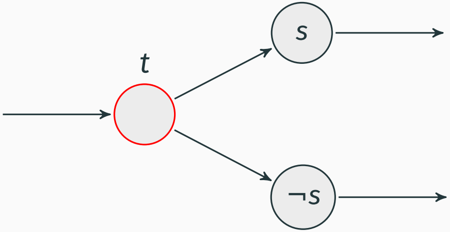


Is F *s* true at *t*?

Branching time



Is F s true at t? Yes.

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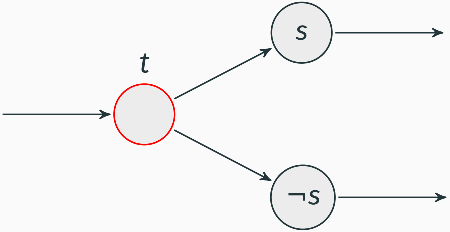
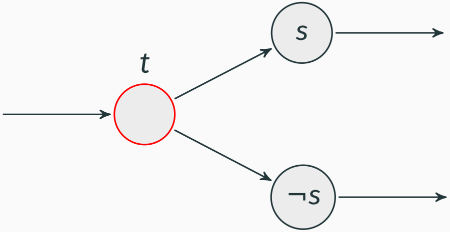


Is F s true at t? Yes.

Is F $\neg s$ true at t?

Branching time



Is F s true at t? Yes.

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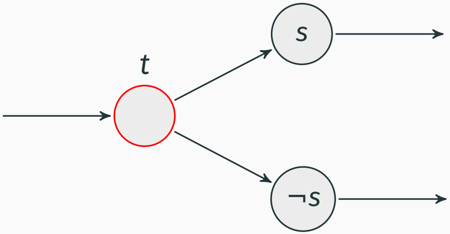
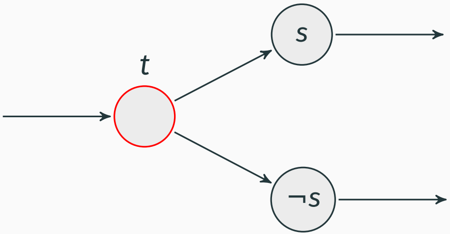


Is F s true at t? Yes.

Is F $\neg s$ true at t? Yes.

Intuition: 'There will be a sea battle' is not true at t.

Branching time



Standard semantics:

$M, t \models \mathsf{F}A$ iff $M, r \models A$ for some r such that $t < r$

iff A is true at some future point on some history through t.

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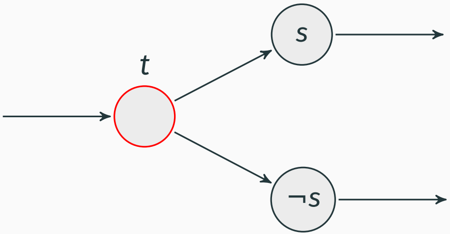

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$M, t \models \mathsf{F} A$ iff A is true at some future point on every history through t.

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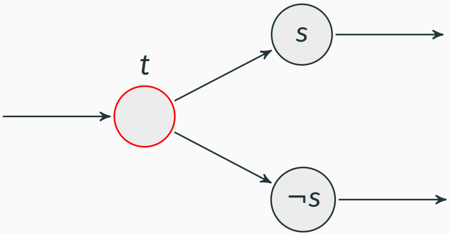

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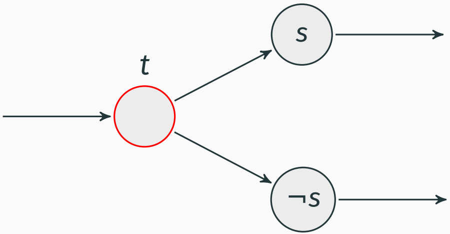
This makes $\mathsf{F}\, s$ and $\mathsf{F}\, \neg s$ both false at t.

Branching time



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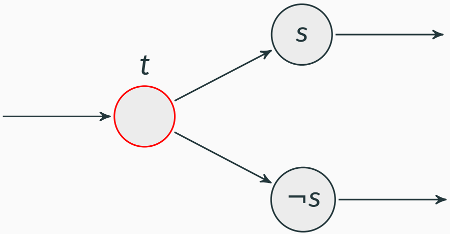

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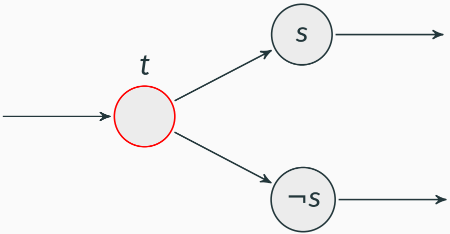

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"Ockhamist" semantics (CTL*):

$M, h, t \models \mathrm{F}A$ iff A is true at some future point on history h through t.

$\mathrm{F}s$ is true at t on the upper history, and false on the lower history.

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$M, h, t \models \Diamond A$ iff A is true at t on some history through t.

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$M, h, t \models \Diamond A$ iff A is true at t on some history through t.
$M, h, t \models \Diamond \mathsf{F} A$ iff A is true at some future point on some history through t.
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- In Ockhamist semantics, truth is defined relative to three **parameters**: M, h, t.

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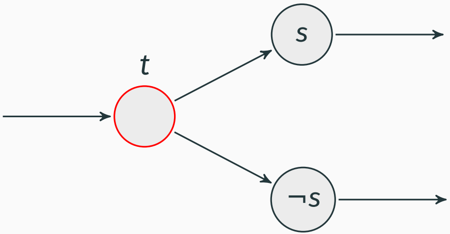
Supervaluationism:

$M, t \models A$ iff $M, h, t \models A$ for all histories h through t.

Branching time



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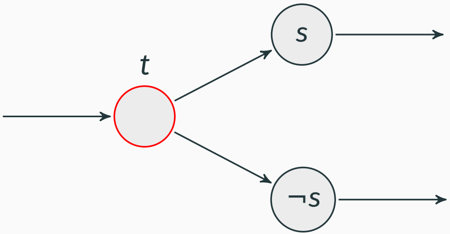

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In supervaluationist Ockhamism,

- $\mathsf{F}s$ and $\mathsf{F}\neg s$ are not true at t.

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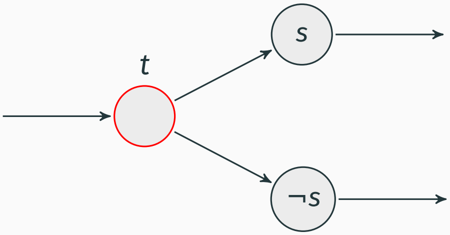

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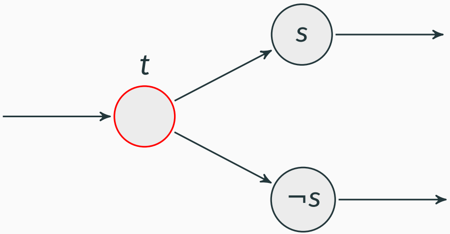

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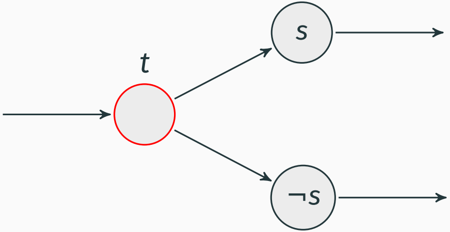
Branching time



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In supervaluationist Ockhamism,

- $\mathrm{F}s$ and $\mathrm{F}\neg s$ are not true at t.
- $\neg\,\mathrm{F}s$ and $\neg\,\mathrm{F}\neg s$ are not true at t.
- $\mathrm{F}s$ and $\mathrm{F}\neg s$ are neither true nor false at t.
- $\mathrm{F}s \vee \neg\,\mathrm{F}s$ is true at t.

Branching time

If we look at possible scenarios, supervaluationist Ockhamism determines a **three-valued logic**: in any given scenario, a sentence can be

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The truth-value of a truth-functionally complex sentence at a scenario is not determined by the truth-value of the parts:

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If we look at possible scenarios, supervaluationist Ockhamism determines a **three-valued logic**: in any given scenario, a sentence can be

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The truth-value of a truth-functionally complex sentence at a scenario is not determined by the truth-value of the parts:

- Fs and $\neg Fs$ are neither true nor false, $Fs \lor \neg Fs$ is true.
- Fs and $F\neg\neg s$ are neither true nor false, $Fs \lor F\neg\neg s$ is neither true nor false.

Branching time

In other three-valued logics, the truth-value of truth-functionally complex sentences is determined by the truth-values of the parts:

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A	B	A ∨ B
1	1	1
1	N	1
1	0	1
N	1	1
N	N	N
N	0	N
0	1	1
0	N	N
0	0	0

Branching time

Another application of supervaluationism: **vagueness**

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Branching time

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- Suppose *F* is a vague predicate, and *a* is a borderline case.
- We can assign a truth-value to *Fa* relative to different **sharpenings** of *F*.
- $M, w, s \models Fa$
- But scenarios (and interpretations) don't include a sharpening.
- $M, w \models Fa$ iff $M, w, s \models Fa$ for all sharpenings s.

Branching time

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Branching time

Another application of supervaluationism: **vagueness**

- Suppose *F* is a vague predicate, and *a* is a borderline case.
- *Fa* is not true.

Branching time

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- Suppose F is a vague predicate, and a is a borderline case.
- Fa is not true.
- $\neg Fa$ is not true.

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Another application of supervaluationism: **vagueness**

- Suppose F is a vague predicate, and a is a borderline case.
- Fa is not true.
- $\neg Fa$ is not true.
- $Fa \vee \neg Fa$ is true.

Two-Dimensional Modal Logic

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(*) Bob already knew yesterday that there would be a talk today.

Let Y mean 'yesterday'.

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- If $\mathrm{K}_b\, p$ is true yesterday, then p is true yesterday.

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- So $\mathrm{Y}\,\mathrm{K}_b\, p$ entails that p is true yesterday.

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(*) Bob already knew yesterday that there would be a talk today.

Let N mean ‘today’ (or ‘now’).

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Translation: $\mathrm{Y}\,\mathrm{K}_b\,\mathrm{N}\,p$.

What is the semantics of N?

Two-Dimensional Modal Logic

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This doesn't work: $M, t \models \mathrm{N} A$ iff $M, t \models A$.

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This doesn't work: $M, t \models \mathrm{N}\, A$ iff $M, t \models A$.

We want $\mathrm{N}\, p$ to be true yesterday iff p is true today.

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Semantics of N

$M, t_0, t \models \mathsf{N}\, A$ iff $M, t_0, t_0 \models A$.

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- If $\mathrm{N}\, p$ is true at t, t^-, then p is true at t, t.
- So if $\mathrm{Y}\, \mathrm{K}_b\, \mathrm{N}\, p$ is true at t, t then p is true at t, t.

Two-Dimensional Temporal Semantics

(a)	$M, t_0, t \models \rho$	iff $V(\rho, t) = 1$.
(b)	$M, t_0, t \models \neg A$	iff $M, t_0, t \not\models A$.
(c)	$M, t_0, t \models A \wedge B$	iff $M, t_0, t \models A$ and $M, t_0, t \models B$.
(d)	$M, t_0, t \models A \vee B$	iff $M, t_0, t \models A$ or $M, t_0, t \models B$.
(g)	$M, t_0, t \models \mathrm{F}\,A$	iff $M, s \models A$ for some $s \in T$ such that $t < s$.
(h)	$M, t_0, t \models \mathrm{G}\,A$	iff $M, s \models A$ for all $s \in T$ such that $t < s$.
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Scenarios don't have two times. When is a sentence true at a time in a model?

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Scenarios don't have two times. When is a sentence true at a time in a model?

Answer: $M, t \models A$ iff $M, t, t \models A$.

An interesting consequence:

Two-Dimensional Modal Logic

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- $\models \mathrm{N}\, p \rightarrow p$

Two-Dimensional Modal Logic

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- $\models \mathrm{N}\, p \rightarrow p$
- $\not\models \mathrm{G}(\mathrm{N}\, p \rightarrow p)$

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- $\models \mathrm{N}\, p \rightarrow p$
- $\not\models \mathrm{G}(\mathrm{N}\, p \rightarrow p)$

Two-Dimensional Modal Logic

An interesting consequence:

- $\models \mathrm{N}\,p \rightarrow p$
- $\not\models \mathrm{G}(\mathrm{N}\,p \rightarrow p)$

So there are logical truths that will become false in the future!