Review: first-order predicate logic
Sentences of first-order predicate logic ($\mathcal{L}_P$) are made up of

- **predicates** $F, G, H, \ldots$,
- **names** $a, b, c, \ldots$,
- **variables** $x, y, z, \ldots$,
- the logical symbols $\neg, \land, \lor, \rightarrow, \leftrightarrow, \forall, \exists$,
- and the parentheses ( and ).
Atomic sentences are formed by conjoining predicates with names or variables:

- $Fa$
- $Gx$
- $Hxy$
- $Jaxy$
If you take an English sentence that contains no logical operators, and you remove all names, you get a predicate.

- 'Bob sits.' ⇒ '– sits'
- 'Bob sees Carol.' ⇒ '– sees –'
- 'Bob is in Rome.' ⇒ '– is in –'
- 'Bob is Carol’s father.' ⇒ '– is –’s father’
- 'Bob meets Carol’s father in Rome.' ⇒ '– meets –’s father in –’
When you translate from English, translate English names as $L_P$-names, and English predicates as $L_P$-predicates.

- ’Bob sits.’ $\Rightarrow Fa$  
  ($a$: Bob, $F$: – sits)

- ’Bob sees Carol.’ $\Rightarrow Rbc$  
  ($b$: Bob, $c$: Carol, $R$: – sees –)

- ’Bob is in Rome.’ $\Rightarrow lbr$  
  ($b$: Bob, $r$: Rome, $l$: – is in –)

- ’Bob is Carol’s father.’ $\Rightarrow Fbc$  
  ($b$: Bob, $c$: Carol, $F$: – is –’s father)

- ’Bob meets Carol’s father in Rome.’ $\Rightarrow Mbcr$  
  ($b$: Bob, $c$: Carol, $r$: Rome, $M$: – meets –’s father in –)
If $A$ and $B$ are sentences, then so are

- $\neg A$,
- $(A \land B)$,
- $(A \lor B)$,
- $(A \rightarrow B)$,
- $(A \leftrightarrow B)$,
- $\forall x A$,
- $\exists x A$, 
- $\exists x A$, 

• ’Bob doesn’t sit.’ ⇒ ¬Sb  (b: Bob, S: – sits)
• ’Bob sits and Alice runs.’ ⇒ Sb ∧ Ra  (a: Alice, b: Bob, S: – sits, R: – runs)
• ’If Bob sits, Alice runs.’ ⇒ Sb → Ra  (a: Alice, b: Bob, S: – sits, R: – runs)
• ’If Bob had sat, Alice would have run.’ ⇒ Hba
  (a: Alice, b: Bob, H: If – had sat, – would have run)
∀x, ∀y, ...and ∃x, ∃y, ...are quantifiers.

Roughly, ∀x means ‘everything/everyone is such that’;

Roughly, ∃x means ‘something/someone is such that’.

- ‘Everyone sits.’ ⇒ ‘Everyone is such that they sit’. ⇒ ∀xFx
- ‘Bob sees somethingi.’ ⇒ ‘Something is such that Bob sees it’. ⇒ ∃xSbx
- ‘Everyone sees someone.’ ⇒ ‘Everyone₁ is such that someone₂ is such that they₁ see them₂’. ⇒ ∀x∃ySxy
- ‘Everyone sees everyone.’ ⇒ ‘Everyone₁ is such that everyone₂ is such that they₁ see them₂’. ⇒ ∀x∀ySxy
Review: first-order predicate logic

Variables $x, y, z \ldots$ function like pronouns (‘it’, ‘they’).

Variables are logical expressions.

When translating from English, you cannot give a meaning to a variable.

Wrong:

- ’Every tiger sleeps.’ $\Rightarrow \forall x Sx$ ($x$: tiger, $S$: – sleeps)
- ’Bob always sleeps.’ $\Rightarrow \forall x Sx$ ($x$: Bob, $S$: – sleeps)
- ’Bob always sleeps.’ $\Rightarrow \forall b Sb$ ($b$: Bob, $S$: – sleeps)
Review: first-order predicate logic

English quantifiers are usually restricted: ‘every tiger’, ‘some student’, ‘a car’, ‘no cat’.

- ‘Every tiger sleeps.’ $\Rightarrow$ ‘Everything is such that if it is a tiger then it sleeps.’
  $\Rightarrow \forall x(Tx \rightarrow Sx)$
- ‘Some student pays attention.’ $\Rightarrow$ ‘Something is such that it is a student and pays attention.’
  $\Rightarrow \exists x(Sx \land Px)$
- ‘A car drove by.’ $\Rightarrow \exists x(Cx \land Dx)$
In $\forall x (Fx \land Gy) \rightarrow Gx$,

- $\forall x$ binds $x$,
- the first two occurrences of $x$ are **bound**,
- the third is **free**,
- $y$ only has a **free** occurrence.

English sentences are never translated into sentences with free variables.
The standard language $\mathcal{L}_{MP}$ of Modal Predicate Logic is the language of predicate logic with the addition of two one-place sentence operators $\Box$ and $\Diamond$.

If $A$ is a sentence, then so are

- $\Box A$, and
- $\Diamond A$. 
In modal predicate logic, we can “look inside” the proposition letters of $\mathcal{L}_M$.

Necessarily, all myriapods are oviparous.
Necessarily, some arthropods are myriapods.

Necessarily, some arthropods are oviparous.

$\mathcal{L}_M$: $\Box p, \Box q \vdash \Box r$

$\mathcal{L}_{MP}$: $\Box \forall x (Fx \rightarrow Gx), \Box \exists x (Hx \landFx) \vdash \Box \exists x (Hx \land Gx)$
But we can do more.

Let $F$ mean ‘– win the lottery’.

- $\forall x \Diamond Fx$
- $\Diamond \forall x Fx$
-modal predicate logic

$\forall x Fx$ is de dicto: it assert of a proposition $(\forall x Fx)$ that it is possible.

$\forall x \Box Fx$ is de re: is attributes a modal property to certain things.
A sentence is **de re** whenever if it has a sub-sentence $\Box A$ or $\Diamond A$ that contains a free variable.

- $\forall x (Fx \to \Box Gx)$
- $\Box \exists x \Box (\forall y (Fy \to Fx))$
- $\Diamond \forall x Fx \to Fa$
- $\Box (\forall x Fx \to \forall y Gy)$
- $\Diamond Fx$
Modal Predicate Logic

English sentences are often ambiguous between a de re and a de dicto reading.

• Every bachelor must be married.
  • $\forall x (Bx \rightarrow \Box Mx)$
  • $\Box \forall x (Bx \rightarrow Mx)$

• Everyone in this room might have stolen the jewels.
  • $\forall x (lxr \rightarrow \Diamond Sxj)$
  • $\Diamond \forall x (lxr \rightarrow Sxj)$
Identity
It is often useful to add a special **identity predicate** to (modal) predicate logic. We write $a = b$ instead of $= ab$, and $a \neq b$ instead of $\neg (a = b)$.

‘$=$’ is a logical predicate. It always means ‘$-$ is (numerically) identical to $-$’.
Identity

Leibniz’s Law:

Hesperus is visible in the evening sky.
Hesperus = Phosphorus.

Therefore: Phosphorus is visible in the evening sky.
Leibniz’s Law:

\[ b = c \]

Here \( A[c//b] \) is the sentence \( A \) with some or all occurrences of \( b \) replaced by \( c \).
A problem:

Hammurabi knows that Hesperus is visible in the evening sky.  
Hesperus = Phosphorus.  

Therefore: Hammurabi knows that Phosphorus is visible in the evening sky.

\[ \Box \neg h \]
\[ h = p \]
\[ \therefore \neg p \]
Identity

Identity is useful not just to express claims about identity:

- $R$ is strongly connected. \( \Rightarrow \forall x \forall y (Rxy \lor x = y \lor Ryx) \)
- Bob exists. \( \Rightarrow \exists x (x = b) \)
- The current Prime Minister is trustworthy. \( \Rightarrow \exists x (Px \land \forall y (Py \rightarrow y = x) \land Tx) \)
Identity

‘The current Prime Minister might have been trustworthy’ is ambiguous.

- $\diamond \exists x (P_x \land \forall y (P_y \rightarrow y = x) \land T_x)$
- $\exists x (P_x \land \forall y (P_y \rightarrow y = x) \land \diamond T_x)$