Logic 2: Modal Logic

Lecture 17

Wolfgang Schwarz

11 November 2019

University of Edinburgh

Review: first-order predicate logic

Review: first-order predicate logic

Sentences of first-order predicate logic ($\mathfrak{L}_P$) are made up of

- **predicates** $F, G, H, \ldots,$
- **names** $a, b, c, \ldots,$
- **variables** $x, y, z, \ldots,$
- the logical symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$,
- and the parentheses (and).

Review: first-order predicate logic

Atomic sentences are formed by conjoining predicates with names or variables:

- Fa
- Gx
- Hxy
- $Jaxy$

Review: first-order predicate logic

If you take an English sentence that contains no logical operators, and you remove all names, you get a predicate.

- 'Bob sits.' ⇒ '– sits'
- 'Bob sees Carol.' ⇒ '– sees –'
- 'Bob is in Rome.' ⇒ '– is in –'
- 'Bob is Carol's father.' ⇒ '– is –'s father'
- 'Bob meets Carol's father in Rome.' ⇒ '– meets –'s father in –'

Review: first-order predicate logic

When you translate from English, translate English names as $\mathfrak{L}_P$-names, and English predicates as $\mathfrak{L}_P$-predicates.

- 'Bob sits.' $\Rightarrow$ Fa (a: Bob, F: – sits)
- 'Bob sees Carol.' $\Rightarrow$ Rbc (b: Bob, c: Carol, R: – sees –)
- 'Bob is in Rome.' $\Rightarrow$ Ibr (b: Bob, r: Rome, I: – is in –)
- 'Bob is Carol's father.' $\Rightarrow$ Fbc (b: Bob, c: Carol, F: – is –'s father)
- 'Bob meets Carol's father in Rome.' $\Rightarrow$ $Mbcr$
  (b: Bob, c: Carol, r: Rome, M: – meets –'s father in –)

Review: first-order predicate logic

If A and B are sentences, then so are

- $\neg A$,
- $(A \wedge B)$,
- $(A \vee B)$,
- $(A \rightarrow B)$,
- $(A \leftrightarrow B)$.
- $\forall x A$,
- $\exists x A$,

Review: first-order predicate logic

- 'Bob doesn't sit.' $\Rightarrow \neg Sb$ (b: Bob, S: – sits)
- 'Bob sits and Alice runs.' $\Rightarrow Sb \wedge Ra$ (a: Alice, b: Bob, S: – sits, R: – runs)
- 'If Bob sits, Alice runs.' $\Rightarrow Sb \rightarrow Ra$ (a: Alice, b: Bob, S: – sits, R: – runs)
- 'If Bob had sat, Alice would have run.' $\Rightarrow Hba$
  (a: Alice, b: Bob, H: If – had sat, – would have run)

Review: first-order predicate logic

$\forall x$, $\forall y$, ...and $\exists x$, $\exists y$, ...are **quantifiers**.

Roughly, $\forall x$ means ‘everything/everyone is such that’;

Roughly, $\exists x$ means ‘something/someone is such that’.

- ‘Everyone sits.’ ⇒ ‘Everyone is such that they sit’. ⇒ $\forall xFx$
- ‘Bob sees somethingi.’ ⇒ ‘Something is such that Bob sees it’. ⇒ $\exists xSbx$
- ‘Everyone sees someone.’ ⇒ ‘Everyone$_1$ is such that someone$_2$ is such that they$_1$ see them$_2$’. ⇒ $\forall x \exists y Sxy$
- ‘Everyone sees everyone.’ ⇒ ‘Everyone$_1$ is such that everyone$_2$ is such that they$_1$ see them$_2$’. ⇒ $\forall x \forall y Sxy$

Review: first-order predicate logic

Variables $x, y, z \ldots$ function like pronouns ('it', 'they').

Variables are logical expressions.

When translating from English, you cannot give a meaning to a variable.

Wrong:

- 'Every tiger sleeps.' $\Rightarrow \forall x S x \quad (x\text{: tiger}, S\text{: – sleeps})$
- 'Bob always sleeps.' $\Rightarrow \forall x S x \quad (x\text{: Bob}, S\text{: – sleeps})$
- 'Bob always sleeps.' $\Rightarrow \forall b S b \quad (b\text{: Bob}, S\text{: – sleeps})$

Review: first-order predicate logic

English quantifiers are usually restricted: ‘every tiger’, ‘some student’, ‘a car’, ‘no cat’.

- ‘Every tiger sleeps.’ $\Rightarrow$ ‘Everything is such that if it is a tiger then it sleeps.’ $\Rightarrow \forall x(Tx \rightarrow Sx)$
- ‘Some student pays attention.’ $\Rightarrow$ ‘Something is such that it is a student and pays attention.’ $\Rightarrow \exists x(Sx \wedge Px)$
- ‘A car drove by.’ $\Rightarrow \exists x(Cx \wedge Dx)$

Review: first-order predicate logic

In $\forall x(Fx \land Gy) \to Gx$,

- $\forall x$ **binds** x,
- the first two occurrences of x are **bound**,
- the third is **free**,
- y only has a **free** occurrence.

English sentences are never translated into sentences with free variables.

Modal Predicate Logic

Modal Predicate Logic

The **standard language** $\mathfrak{L}_{MP}$ **of Modal Predicate Logic** is the language of predicate logic with the addition of two one-place sentence operators $\Box$ and $\Diamond$.

If A is a sentence, then so are

- $\Box A$, and
- $\Diamond A$.

Modal Predicate Logic

In modal predicate logic, we can "look inside" the proposition letters of $\mathfrak{L}_M$.

Necessarily, all myriapods are oviparous.
Necessarily, some arthropods are myriapods.

Necessarily, some arthropods are oviparous.

$\mathfrak{L}_M$: $\quad \Box p, \Box q \therefore \Box r$

$\mathfrak{L}_{MP}$: $\quad \Box\forall x(Fx \rightarrow Gx), \Box\exists x(Hx \wedge Fx) \therefore \Box\exists x(Hx \wedge Gx)$

Modal Predicate Logic

But we can do more.

Let F mean '– win the lottery'.

- $\forall x \Diamond Fx$
- $\Diamond \forall x Fx$

Modal Predicate Logic

$\Diamond \forall x Fx$ is **de dicto**: it assert of a proposition ($\forall x Fx$) that it is possible.

$\forall x \Diamond Fx$ is **de re**: is attributes a modal property to certain things.

Modal Predicate Logic

A sentence is **de re** whenever if it has a sub-sentence $\Box A$ or $\Diamond A$ that contains a free variable.

- $\forall x(Fx \rightarrow \Box Gx)$
- $\Box\exists x\Box(\forall y(Fy \rightarrow Fx))$
- $\Diamond\forall xFx \rightarrow Fa$
- $\Box(\forall xFx \rightarrow \forall yGy)$
- $\Diamond Fx$

Modal Predicate Logic

English sentences are often ambiguous between a de re and a de dicto reading.

- Every bachelor must be married.
  - $\forall x(Bx \rightarrow \Box Mx)$
  - $\Box\forall x(Bx \rightarrow Mx)$
- Everyone in this room might have stolen the jewels.
  - $\forall x(Ixr \rightarrow \Diamond Sxj)$
  - $\Diamond\forall x(Ixr \rightarrow Sxj)$

Identity

Identity

It is often useful to add a special **identity predicate** to (modal) predicate logic.

We write $a = b$ instead of $= ab$, and $a \neq b$ instead of $\neg(a = b)$.

'$=$' is a logical predicate. It always means '– is (numerically) identical to –'.

Identity

Leibniz's Law:

Hesperus is visible in the evening sky.
Hesperus = Phosphorus.

Therefore: Phosphorus is visible in the evening sky.

Identity

Leibniz's Law:

$$\frac{\begin{array}{l} A \\ b = c \end{array}}{A[c//b]}$$

Here $A[c//b]$ is the sentence A with some or all occurrences of b replaced by c.

Identity

A problem:

Hammurabi knows that Hesperus is visible in the evening sky.
Hesperus = Phosphorus.

Therefore: Hammurabi knows that Phosphorus is visible in the evening sky.

$\Box Vh$
$h = p$

$\Box Vp$

Identity

Identity is useful not just to express claims about identity:

- R is strongly connected. ⇒ $\forall x \forall y(Rxy \lor x{=}y \lor Ryx)$
- Bob exists. ⇒ $\exists x(x{=}b)$
- The current Prime Minister is trustworthy. ⇒ $\exists x(Px \land \forall y(Py \rightarrow y{=}x) \land Tx)$

Identity

'The current Prime Minister might have been trustworthy' is ambiguous.

- $\Diamond\exists x(Px \land \forall y(Py \rightarrow y{=}x) \land Tx)$
- $\exists x(Px \land \forall y(Py \rightarrow y{=}x) \land \Diamond Tx)$