Logic 2: Modal Logic

Lecture 18

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Trees for first-order predicate logic

Trees for first-order predicate logic

Target: $\exists x(Fx \wedge Gx) \rightarrow \exists xFx$

1. $\neg(\exists x(Fx \wedge Gx) \rightarrow \exists xFx)$ (Ass.)

Trees for first-order predicate logic

Target: $\exists x(Fx \wedge Gx) \rightarrow \exists xFx$

1.	$\neg(\exists x(Fx \wedge Gx) \rightarrow \exists xFx)$	(Ass.)
2.	$\exists x(Fx \wedge Gx)$	(1)
3.	$\neg\exists xFx$	(1)

Trees for first-order predicate logic

Target: $\exists x(Fx \wedge Gx) \rightarrow \exists x Fx$

1.	$\neg(\exists x(Fx \wedge Gx) \rightarrow \exists x Fx)$	(Ass.)
2.	$\exists x(Fx \wedge Gx)$	(1)
3.	$\neg\exists x Fx$	(1)
4.	$Fa \wedge Ga$	(2)

Trees for first-order predicate logic

Target: $\exists x(Fx \wedge Gx) \rightarrow \exists xFx$

1.	$\neg(\exists x(Fx \wedge Gx) \rightarrow \exists xFx)$	(Ass.)
2.	$\exists x(Fx \wedge Gx)$	(1)
3.	$\neg\exists xFx$	(1)
4.	$Fa \wedge Ga$	(2)
5.	Fa	(4)
6.	Ga	(4)

Trees for first-order predicate logic

Target: $\exists x(Fx \land Gx) \to \exists x Fx$

1.	$\neg(\exists x(Fx \land Gx) \to \exists x Fx)$	(Ass.)
2.	$\exists x(Fx \land Gx)$	(1)
3.	$\neg \exists x Fx$	(1)
4.	$Fa \land Ga$	(2)
5.	Fa	(4)
6.	Ga	(4)
7.	$\neg Fa$	(3)
	x	

Trees for first-order predicate logic

$\forall x A$

$A[c/x]$

↑ old or first

$\exists x A$

$A[c/x]$

↑ new

$\neg\forall x A$

$\neg A[c/x]$

↑ new

$\neg\exists x A$

$\neg A[c/x]$

↑ old or first

Existence:

$c = c$

↑ old

Leibniz' Law:

$b = c$

A

$A[c//b]$

Trees for first-order predicate logic

Target: $\forall x \forall y((Rxy \land x = y) \rightarrow Rxx)$

1. $\neg \forall x \forall y((Rxy \land x = y) \rightarrow Rxx)$ (Ass.)

Trees for first-order predicate logic

Target: $\forall x \forall y((Rxy \land x = y) \to Rxx)$

1. $\neg\forall x \forall y((Rxy \land x = y) \to Rxx)$ (Ass.)
2. $\neg\forall y((Ray \land a = y) \to Raa)$ (1)

Trees for first-order predicate logic

Target: $\forall x \forall y((Rxy \land x{=}y) \rightarrow Rxx)$

1.	$\neg\forall x \forall y((Rxy \land x{=}y) \rightarrow Rxx)$	(Ass.)
2.	$\neg\forall y((Ray \land a{=}y) \rightarrow Raa)$	(1)
3.	$\neg((Rab \land a{=}b) \rightarrow Raa)$	(2)

Trees for first-order predicate logic

Target: $\forall x \forall y((Rxy \wedge x{=}y) \rightarrow Rxx)$

1.	$\neg\forall x \forall y((Rxy \wedge x{=}y) \rightarrow Rxx)$	(Ass.)
2.	$\neg\forall y((Ray \wedge a{=}y) \rightarrow Raa)$	(1)
3.	$\neg((Rab \wedge a{=}b) \rightarrow Raa)$	(2)
4.	$Rab \wedge a{=}b$	(3)
5.	$\neg Raa$	(3)

Target: $\forall x \forall y((Rxy \wedge x = y) \rightarrow Rxx)$

1.	$\neg \forall x \forall y((Rxy \wedge x = y) \rightarrow Rxx)$	(Ass.)
2.	$\neg \forall y((Ray \wedge a = y) \rightarrow Raa)$	(1)
3.	$\neg((Rab \wedge a = b) \rightarrow Raa)$	(2)
4.	$Rab \wedge a = b$	(3)
5.	$\neg Raa$	(3)
6.	Rab	(4)
7.	$a = b$	(4)

Trees for first-order predicate logic

Target: $\forall x \forall y((Rxy \land x{=}y) \to Rxx)$

1.	$\neg\forall x \forall y((Rxy \land x{=}y) \to Rxx)$	(Ass.)
2.	$\neg\forall y((Ray \land a{=}y) \to Raa)$	(1)
3.	$\neg((Rab \land a{=}b) \to Raa)$	(2)
4.	$Rab \land a{=}b$	(3)
5.	$\neg Raa$	(3)
6.	Rab	(4)
7.	$a{=}b$	(4)
8.	Raa	(6,7,LL)
	x	

Semantics of predicate logic, I

Semantics of predicate logic, I

Sentences of predicate logic are true or false relative to a model.

A **model of first-order predicate logic** consists of

- a non-empty set D (the “domain of individuals”), and
- an interpretation V of the non-logical vocabulary.

Semantics of predicate logic, I

The non-logical vocabulary of $\mathfrak{L}_P$ are the names and the predicates.

Intuitively:

- *Fa* is true (in a given scenario) iff the individual picked out by '*a*' has the property expressed by '*F*'.
- *Rab* is true iff the individual picked out by '*a*' stands to the individual picked out by '*b*' in the relation expressed by '*R*'.

Semantics of predicate logic, I

So to determine the truth-values of $\mathfrak{L}_P$-sentences in a scenario, we might specify

(a) which individuals are picked out by the names,
(b) which properties and relations are expressed by the predicates, and
(c) which individuals have which properties and stand in which relations to one another.

We can be more economical.

Instead of (b) and (c), we simply specify which predicates apply to which individuals.

Semantics of predicate logic, I

$V(a)$ = Alice
$V(F)$ = { Alice, Bob }

$V(F)$ is the set of individuals to which F applies.

Fa is true because $V(a)$ is a member of $V(F)$.

Semantics of predicate logic, I

$V(a)$ = Alice
$V(b)$ = Bob
$V(F)$ = { Alice, Bob }
$V(R) = \{ \langle Alice, Alice \rangle, \langle Bob, Alice \rangle \}$

Raa is true because $\langle V(a), V(a) \rangle$ is a member of $V(R)$.

Rab is false because $\langle V(a), V(b) \rangle$ is not a member of $V(R)$.

Semantics of predicate logic, I

$V(a)$ = Alice
$V(F)$ = { Alice, Bob }

Is $\forall x F x$ true?

It depends on whether the scenario involves other individuals than Alice and Bob.

So we also need to specify the set of all relevant individuals in the scenario.

Semantics of predicate logic, I

A **(classical) first-order model** is a pair $\langle D, V \rangle$ consisting of

- a non-empty set D, and
- a function V that assigns
  - to each name a member of D,
  - to each 1-place predicate a subset of D,
  - to each n-place predicate ($n > 1$) a set of n-tuples from D.

A fragment of first-order logic

A fragment of first-order logic

Let's look at a fragment $\mathfrak{L}_P^1$ of $\mathfrak{L}_P$ with only 1-place predicates, no names, and only one variable, x.

Atomic sentences: $Fx, Gx, Hx, \ldots$

A **model** of $\mathfrak{L}_P^1$ consists of

- an **(individual) domain** D
- an **interpretation function** V that assigns a subset of D to each predicate.

A fragment of first-order logic

Variables do not have a fixed meaning (even in a model).

If some members of D are F and others $\neg F$, then

- Fx is **true of** some individuals,
- Fx is **false of** other individuals.

$\exists x Fx$ is true iff Fx is true of something (in D).

$\forall x Fx$ is true iff Fx is true of everything (in D).

A fragment of first-order logic

In general: $\forall x A$ is true iff A is true of everything (in D).

Let $M, d \models A$ mean that A is true of d in M.

We need to specify for any M, d, and A whether $M, d \models A$.

A fragment of first-order logic

Semantics of $\mathfrak{L}^1_P$

(a)	$M, d \models Fx$	iff $d \in V(F)$.
(b)	$M, d \models \neg A$	iff $M, d \not\models A$.
(c)	$M, d \models A \land B$	iff $M, d \models A$ and $M, d \models B$.
(d)	$M, d \models A \lor B$	iff $M, d \models A$ or $M, d \models B$.
(e)	$M, d \models A \rightarrow B$	iff $M, d \models B$ or $M, d \not\models A$.
(f)	$M, d \models A \leftrightarrow B$	iff $M, d \models (A \rightarrow B)$ and $M, d \models (B \rightarrow A)$.
(g)	$M, d \models \forall x A$	iff $M, e \models A$ for all $e \in D$.
(h)	$M, d \models \exists x A$	iff $M, e \models A$ for some $e \in D$.

A fragment of first-order logic

A **model** of $\mathfrak{L}^1_P$ consists of

- an **(individual) domain** D
- an **interpretation function** V that assigns a subset of D to each predicate.

(a)	$M, d \models Fx$	iff $d \in V(F)$.
(b)	$M, d \models \neg A$	iff $M, d \not\models A$.
(c)	$M, d \models A \wedge B$	iff $M, d \models A$ and $M, d \models B$.
(d)	$M, d \models A \vee B$	iff $M, d \models A$ or $M, d \models B$.
(e)	$M, d \models A \rightarrow B$	iff $M, d \models B$ or $M, d \not\models A$.
(f)	$M, d \models A \leftrightarrow B$	iff $M, d \models (A \rightarrow B)$ and $M, d \models (B \rightarrow A)$.
(g)	$M, d \models \forall x A$	iff $M, e \models A$ for all $e \in D$.
(h)	$M, d \models \exists x A$	iff $M, e \models A$ for some $e \in D$.

A fragment of first-order logic

A **model** of $\mathfrak{L}^1_P$ consists of

- an **(individual) domain** D
- an **interpretation function** V that assigns to each member of D a truth-value.

(a)	$M, d \models Fx$	iff $V(F, d) = 1$.
(b)	$M, d \models \neg A$	iff $M, d \not\models A$.
(c)	$M, d \models A \wedge B$	iff $M, d \models A$ and $M, d \models B$.
(d)	$M, d \models A \vee B$	iff $M, d \models A$ or $M, d \models B$.
(e)	$M, d \models A \rightarrow B$	iff $M, d \models B$ or $M, d \not\models A$.
(f)	$M, d \models A \leftrightarrow B$	iff $M, d \models (A \rightarrow B)$ and $M, d \models (B \rightarrow A)$.
(g)	$M, d \models \forall x A$	iff $M, e \models A$ for all $e \in D$.
(h)	$M, d \models \exists x A$	iff $M, e \models A$ for some $e \in D$.

A fragment of first-order logic

$\mathfrak{L}^1_P$, as standardly interpreted, is a notational variant of $\mathfrak{L}_M$, as interpreted in chapter 2.

A similar result holds for $\mathfrak{L}_M$, as interpreted in Kripke semantics.

> Modal propositional logic is a fragment of first-order predicate logic.

Semantics of predicate logic, II

Semantics of predicate logic, II

A **(classical) first-order model** is a pair $\langle D, V \rangle$ consisting of

- a non-empty set D, and
- a function V that assigns
  - to each name a member of D,
  - to each 1-place predicate a subset of D,
  - to each n-place predicate ($n > 1$) a set of n-tuples from D.

Semantics of predicate logic, II

$D = \{Alice, Bob, Carol\}$
$V(a) = \text{Alice}$
$V(b) = \text{Bob}$
$V(F) = \{\text{ Alice, Bob }\}$

Is $\forall x \exists y (Fx \vee Fy)$ true?

Fx is true of Alice and Bob, but not Carol.
Fy is true of Alice and Bob, but not Carol.

$Fx \vee Fy$ is true of ...?

$Fx \vee Fy$ is true relative to $[x \mapsto \text{Alice}, y \mapsto \text{Bob}]$
$Fx \vee Fy$ is true relative to $[x \mapsto \text{Alice}, y \mapsto \text{Carol}]$
$Fx \vee Fy$ is not true relative to $[x \mapsto \text{Carol}, y \mapsto \text{Carol}]$

Semantics of predicate logic, II

An arbitrary $\mathfrak{L}_P$-sentence is **true relative to an assignment** g **of individuals to the variables**.

$\forall xFx$ is true iff Fx is true no matter what is assigned to x.

In general:

$\forall xA$ is true iff A is true no matter what is assigned to x.

Semantics of predicate logic, II

A variable assignment g' is an x-**variant** of a variable assignment g iff g' differs from g at most in the value it assigns to x.

$\exists y \forall x Rxy$ is true iff

- there is some assignment g of individuals to variables such that
- for all x-variants g' of g,
- Rxy is true relative to g'.

Semantics of predicate logic, II

$$[t]^{M,g} =_{\text{def}} \begin{cases} \text{V(t)} & \text{if } t \text{ is a name (and } V \text{ is the interpretation function of } M) \\ \text{g(t)} & \text{if } t \text{ is a variable.} \end{cases}$$

Semantics of predicate logic, II

Semantics of first-order logic

(a)	$M, g \models \phi t_1 \ldots t_n$	iff $\langle [t_1]^{M,g}, \ldots, [t_n]^{M,g} \rangle \in V(\phi)$.
(b)	$M, g \models s = t$	iff $[s]^{M,g} = [t]^{M,g}$.
(c)	$M, g \models \neg A$	iff $M, g \not\models A$.
(d)	$M, g \models A \wedge B$	iff $M, g \models A$ and $M, g \models B$.
(e)	$M, g \models A \vee B$	iff $M, g \models A$ or $M, g \models B$.
(f)	$M, g \models A \rightarrow B$	iff $M, g \models B$ or $M, g \not\models A$.
(g)	$M, g \models A \leftrightarrow B$	iff $M, g \models (A \rightarrow B)$ and $M, g \models (B \rightarrow A)$.
(h)	$M, g \models \forall x A$	iff $M, g' \models A$ for all x-variants g' of g.
(i)	$M, g \models \exists x A$	iff $M, g' \models A$ for some x-variant g' of g.