Logic 2: Modal Logic

Lecture 19

Wolfgang Schwarz

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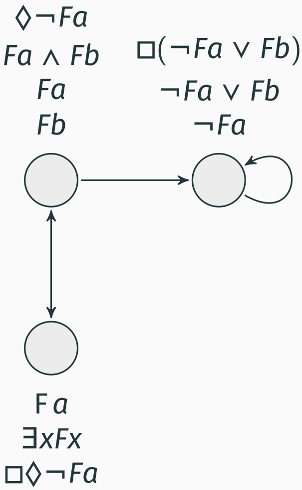
University of Edinburgh

Constant domain semantics

Constant domain semantics



Constant domain semantics

A model must contain enough information to tell us which sentences of $\mathfrak{L}_P$ are true at any world.

To determine the truth-values of $\mathfrak{L}_P$-sentences we need to know

- what the quantifiers range over (a domain),
- for every name, which member of the domain it picks out,
- for every predicate, which members or tuples from the domain it applies to.

In **constant domain semantics**, we assume that all worlds have the same domain.

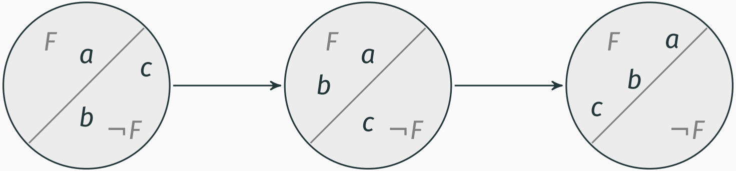
Constant domain semantics

A **constant domain model** consists of

1. a set of worlds W,
2. an accessibility relation R on W,
3. a set of individuals D,
4. an interpretation function V that assigns
   - to each name a member of D,
   - to each n-place predicate and world w a set of n-tuples from D.

Predicates can apply to different things at different worlds.

Constant domain semantics



Fa
$\neg Fb$
$\exists x \Diamond \neg Fx$
$\Box \forall x \Box Fx$

Fa
$\neg Fc$
$\Diamond \forall x Fx$

$\forall x Fx$

Constant domain semantics

Constant domain semantics

(a) $M, w, g \models \phi t_1 \ldots t_n$ iff $\langle [t_1]^{M,g}, \ldots, [t_n]^{M,g} \rangle \in V(\phi, w)$.

(b) $M, w, g \models s = t$ iff $[s]^{M,g} = [t]^{M,g}$.

(c) $M, w, g \models \neg A$ iff $M, w, g \not\models A$.

(d) $M, w, g \models A \wedge B$ iff $M, w, g \models A$ and $M, w, g \models B$.

(e) $M, w, g \models A \vee B$ iff $M, w, g \models A$ or $M, w, g \models B$.

(f) $M, w, g \models A \rightarrow B$ iff $M, w, g \models B$ or $M, w, g \not\models A$.

(g) $M, w, g \models A \leftrightarrow B$ iff $M, w, g \models (A \rightarrow B)$ and $M, w, g \models (B \rightarrow A)$.

(h) $M, w, g \models \forall x A$ iff $M, w, g' \models A$ for all x-variants g' of g.

(i) $M, w, g \models \exists x A$ iff $M, w, g' \models A$ for some x-variant g' of g.

(j) $M, w, g \models \Box A$ iff $M, v, g \models A$ for all $v \in W$ such that wRv.

(k) $M, w, g \models \Diamond A$ iff $M, v, g \models A$ for some $v \in W$ such that wRv.

(l) $M, w \models A$ iff $M, w, g \models A$ for all variable assignments g.

The logic of constant domains

The logic of constant domains

A sentence is **valid in constant domain semantics** (for short **CK**-valid) iff it is true at all worlds in all constant domain models.

- All sentences that are valid in classical predicate logic are CK-valid.
- All K-valid sentences are CK-valid.
- Anything that can be derived by combining the tree rules for classical predicate logic with those of K is CK-valid.

The logic of constant domains

The Barcan Formula:

$$(\textbf{BF}) \quad \forall x \Box A \rightarrow \Box \forall x A$$

BF is CK-valid:

1. Suppose $\forall x \Box Fx$ is true at w.
2. Then every individual is F at every w-accessible world.
3. And then $\Box \forall x Fx$ is true at w.

The logic of constant domains

1. $\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$ (w) (Ass.)

The logic of constant domains

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)

The logic of constant domains

1.	$\neg(\forall x \Box Fx \to \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)

The logic of constant domains

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)
6.	$\neg Fa$	(v)	(5)

The logic of constant domains

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)
6.	$\neg Fa$	(v)	(5)
7.	$\Box Fa$	(w)	(2)

The logic of constant domains

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)
6.	$\neg Fa$	(v)	(5)
7.	$\Box Fa$	(w)	(2)
8.	Fa	(v)	(7,4)
	x		

The logic of constant domains

$$(\textbf{BF}) \quad \forall x \Box A \rightarrow \Box \forall x A$$

BF is invalid iff individuals at other worlds can fail to exist at the actual world.

1. Suppose unicorns could have existed, but nothing that actually exists could have been a unicorn.
2. Then $\forall x \Box \neg Ux$ is true.
3. But $\Box \forall x \neg Ux$ is false.

The logic of constant domains

(**BF**) $\forall x \Box A \rightarrow \Box \forall x A$

(**CBF**) $\Box \forall x A \rightarrow \forall x \Box A$

CBF is also CK-valid.

1. Suppose $\Box \forall x Fx$ is true at w.
2. Then every individual is F at every w-accessible world.
3. And then $\forall x \Box Fx$ is true at w.

The logic of constant domains

(**BF**) $\forall x \Box A \rightarrow \Box \forall x A$

(**CBF**) $\Box \forall x A \rightarrow \forall x \Box A$

CBF is invalid iff some actual individual fails to exist at other worlds.

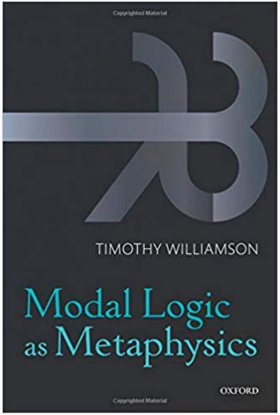
1. Suppose you could have failed to exist.
2. Then $\Box \forall x \exists y (x = y)$ is true.
3. But $\forall x \Box \exists y (x = y)$ is false.

The logic of constant domains

We can also prove $\forall x \Box \exists y(x = y)$.

1.	$\neg \forall x \Box \exists y(x = y)$	(w)	(Ass.)
2.	$\neg \Box \exists y(a = y)$	(w)	(1)
3.	wRv		(2)
4.	$\neg \exists y(a = y)$	(v)	(2)
5.	$\neg (a = a)$	(v)	(4)
6.	$a = a$	(v)	(Ex)
	x		

The logic of constant domains



Necessitism:

- Everything necessarily exists.
- Nothing could have failed to exist.
- If your parents had never met, you would still have existed, but you would not have been a person.

Permanentism:

- Everything has always existed and will always exist.
- Anything that ever existed or will exist exists now.
- The dinosaurs still exist, but they are no longer dinosaurs.

The logic of constant domains

Like in propositional modal logic, we get stronger logics if we add further constraints on the accessibility relation.

- Reflexivity makes the **T**-schema valid.
- Seriality makes the **D**-schema valid.
- Transitivity makes the **4**-schema valid.
- Euclidity makes the **5**-schema valid.
- Convergence makes the **G**-schema valid.

The logic of constant domains

Oddly, completeness results do not always carry over from propositional to predicate logic.

- Adding **T**, **4**, and **G** to the axiomatic calculus for **K** yields a calculus that is complete with respect to reflexive, transitive, and convergent models.
- Adding **T**, **4**, and **G** to the axiomatic calculus for **CK** does not yield a calculus that is complete with respect to reflexive, transitive, and convergent models.

$$\neg(\Diamond(\exists x Ax \land \forall x(Ax \to \Box Bx) \land \Box\neg\forall x Bx) \ \land\ \Diamond\forall x(Ax \lor \Box Bx) \land \forall x(\Diamond Ax \to \Box(\exists x Ax \to Ax)))$$

Variable domain semantics

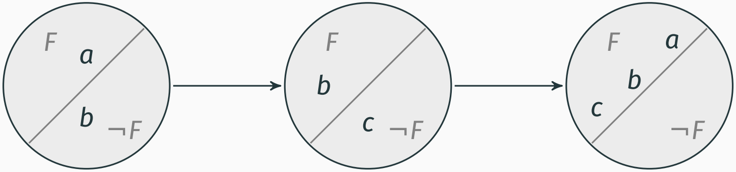
Variable domain semantics

Let's drop the assumption that the very same things exist at every world.

A **variable domain model** consists of

1. a set of worlds W,
2. an accessibility relation R on W,
3. for each world w, a set of individuals D_w,
4. an interpretation function V that assigns
    - to each name a member of D,
    - to each n-place predicate and world w a set of n-tuples from D. (?)

Variable domain semantics



Fa

Fb

$\neg\exists x(x = c)$

Fc ? $\neg Fc$?

$c = c$?

Variable domain semantics

Free logic is classical predicate logic without the assumption that names always refer.

Intuition: It is not a logical truth that Boris Johnson exists.

But $\exists x(x = b)$ is a logical truth.

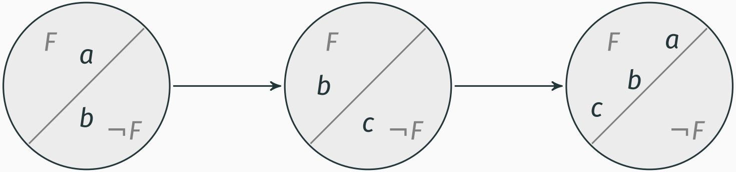
Variable domain semantics

Three kinds of free logic

1. **negative**: If *b* doesn't exist, then all atomic sentences involving *b* are false.
2. **positive**: If *b* doesn't exist, then *Fb* may still be true while *Gb* is false.
3. **non-valent**: If *b* doesn't exist, then atomic sentences involving *b* are neither true nor false.

Variable domain semantics

Variable domains with a negative semantics:



$$Fa$$
$$Fb$$
$$\neg\exists x(x = c)$$
$$\neg Fc$$
$$c \neq c$$

Variable domain semantics

A **negative variable domain model** consists of

1. a set of worlds W,
2. an accessibility relation R on W,
3. for each world w, a set of individuals D_w,
4. an interpretation function V that assigns
   - to each name a member of some D_w,
   - to each n-place predicate and world w a set of n-tuples from D_w.

Variable domain semantics

(a)	$M, w, g \models \phi t_1 \dots t_n$	iff $\langle [t_1]^{M,g}, \dots, [t_n]^{M,g} \rangle \in V(\phi, w)$.
(b)	$M, w, g \models s = t$	iff $[s]^{M,g} = [t]^{M,g}$ and $[s]^{M,g}$ and $[t]^{M,g}$ are in D_w.
(c)	$M, w, g \models \neg A$	iff $M, w, g \not\models A$.
	$\vdots$	
(h)	$M, w, g \models \forall x A$	iff $M, w, g' \models A$ for all x-variants g' of g for which $g'(x) \in D_w$.
(i)	$M, w, g \models \exists x A$	iff $M, w, g' \models A$ for some x-variant g' of g for which $g'(x) \in D_w$.
(j)	$M, w, g \models \Box A$	iff $M, v, g \models A$ for all $v \in W$ such that wRv.
(k)	$M, w, g \models \Diamond A$	iff $M, v, g \models A$ for some $v \in W$ such that wRv.

Variable domain semantics

In variable domain semantics, **BF**, **CBF**, and **NE** are invalid.

$$(\textbf{BF})\quad \forall x \Box A \to \Box \forall x A$$

$$(\textbf{CBF})\quad \Box \forall x A \to \forall x \Box A$$

$$(\textbf{NE})\quad \forall x \Box \exists y (y = x)$$

Their apparent proofs employ the rule of Universal Instantiation:

$\forall x A$
$\therefore A[c/x]$

This rule is invalid in free logic.