Review: Language

Review: Language

Modal logic is widely used to formalise reasoning with **non-truth-functional** concepts.

- knowledge
- belief
- provability
- obligation and permission
- what will or was the case
- what could have been the case
- what would have been the case if so-and-so had been the case
- ...

Review: Language

To reason with these concepts, we add **new sentence operators** to the language of classical propositional or predicate logic.

- $\Box$ – can mean anything, often: it is necessity that
- K – it is known that
- B – it is believed that
- O – it is obligatory that
- G – it is always going to be the case that
- H – it has always been the case that
- $A \mathrel{\Box\!\!\rightarrow} B$ – if A had been the case B would have been the case
- ...

Review: Language

Every modal operator has a **dual**:

$$\Diamond A \iff_{\text{def}} \neg\Box\neg A$$

$A \strictif B$ is equivalent to $\Box(A \rightarrow B)$.

Review: Language

Heuristics for translating from English into the language of modal logic:

- First paraphrase the original English sentence with sentence operators like ‘it is necessary that’, ‘it is possible that’, ‘it is required that’.
- Translate sentence letters by full sentences without any relevant logical expressions.
- Check if you can think of a scenario in which your translation and the original sentence have different truth-values. Try edge cases!
- Read $A \rightarrow B$ as ‘not A or B’.
- Avoid $A \rightarrow \Box B$.
- Avoid $\Diamond(A \rightarrow B)$.

(a) We may have to pay a fine.

Paraphrase:

- It is (epistemically) possible that it is required that we pay a fine.

Translation:

- $\Diamond \mathrm{O}\, p$
  p: We pay a fine.

Review: Language

(b) If the sky is clear, it can't be raining.

Paraphrase:

- If the sky is clear, then it is not (epistemically) possible that it is raining.

Translation:

- $p \rightarrow \neg\Diamond q$?
  p: the sky is clear
  q: it is raining

$p \rightarrow \neg\Diamond q$ is false whenever p is true and $\neg\Diamond q$ is false.

Is there a scenario in which $\Diamond q$ is true, p is true, and yet (b) is true? Yes!

Review: Language

(b) If the sky is clear, it can't be raining.

Paraphrase:

- It is certain that if the sky is clear, then it is not raining.
- It is not possible that the sky is clear and it is raining.

Translation:

- $\Box(p \rightarrow \neg q)$
  p: the sky is clear
  q: it is raining
- $\neg\Diamond(p \wedge q)$

(c) The dog must not be left alone in the home for longer than 5 hours or overnight.

Paraphrase:

- It is not allowed that the dog is left alone in the home for longer than 5 hours or the dog is left alone in the home overnight.

Translation:

- $\neg \mathrm{P}(p \vee q)$
  p: The dog is left alone in the home for longer than 5 hours.
  q: The dog is left alone in the home overnight.
- $\mathrm{O} \neg (p \vee q)$
- $\mathrm{O} \neg p \wedge \mathrm{O} \neg q$

(d) A nuclear strike might lead to many innocent deaths and so can't be the right thing to do.

- $\Diamond p \wedge \neg \Diamond \mathrm{O}\, q$
  p: A nuclear strike leads to many innocent deaths.
  q: A nuclear strike is carried out
- Close: $\mathrm{O}(\neg q / \Diamond p)$, but misses 'cannot be', and fails to imply $\Diamond p$.
- Wrong: $(q \rightarrow \Diamond r) \rightarrow \neg \mathrm{P}\, q$.
  q: nuclear strike
  r: innocent deaths

  This is equivalent to $(\neg q \rightarrow \neg \mathrm{P}\, q) \wedge (\Diamond r \rightarrow \neg \mathrm{P}\, q)$.

Exam question 1:

Translate the following sentences, as well as possible, into a suitable modal language. (The resources of modal predicate logic are only needed for d.)

(a) You can keep your shoes on.
(b) I have never been to Italy.
(c) ...
(d) ...

Review: Models

Review: Models

The main task of a logic is to settle which sentences are (logically) **entailed** by which sentences.

Informally, A logically entails B iff there is no conceivable scenario in which A is true and B is false, on any interpretation of the non-logical vocabulary.

Formally, we represent conceivable scenarios and interpretations of the non-logical vocabulary by a **model**.

$A \models B$ iff B is true in every model in which A is true.

Review: Models

In most logics, $A \models B$ iff $\models A \rightarrow B$.

Review: Models

To clarify the logic of modal operators, we often use **possible-worlds models**.

Guiding intuition:

- *p* is possible iff *p* is true at some (relevantly) possible world.
- *p* is necessary iff *p* is true at every (relevantly) possible world.

Review: Models

A **Kripke model** for modal propositional logic consists of

- a set of "worlds" W,
- a binary "accessibility" relation R on W, and
- an interpretation function V that assigns to each sentence letter of $\mathfrak{L}_M$ and each member of W a truth-value.

Kripke semantics for modal propositional logic specifies, for any world w in any Kripke model M, and any $\mathfrak{L}_M$-sentence A, whether A is true at w in M.

Review: Models

The **accessibility relation** in Kripke model represents different things, depending on the application.

- wRv iff v is compatible with the laws of physics at w.
- wRv iff v is compatible with the knowledge at w.
- wRv iff v is compatible with the norms at w.
- wRv iff v is compatible with the essence of things.
- wRv iff v is later than w.
- ...

Review: Models

Different interpretations of accessibility impose different formal constraints on Kripke models.

- The actual world is always compatible with our knowledge. So in epistemic models, every world is accessible from itself.
- The actual world is not always compatible with the norms. So in deontic models, some worlds may not be accessible from themselves.

Review: Models

Different interpretations of accessibility impose different formal constraints on Kripke models.

- R is **reflexive** if every world is accessible from itself.
- R is **serial** if every world can access some world.
- R is **universal** if every world can access every world.
- R is **transitive** if whenever wRv and vRu then wRu.
- R is **euclidean** if whenever wRv and wRu then vRu.
- ...

(You don't need to memorize which is which.)

Review: Models

By imposing constraints on Kripke models, we get different **logics** (or **systems**).

- A sentence is **K-valid** if it is true at all worlds in all Kripke models.
- A sentence is **T-valid** if it is true at all worlds in all Kripke models in which R is reflexive.
- A sentence is **D-valid** if it is true at all worlds in all Kripke models in which R is serial.
- A sentence is **S5-valid** if it is true at all worlds in all Kripke models in which R is universal.
- A sentence is **S4-valid** if it is true at all worlds in all Kripke models in which R is reflexive and transitive.
- ...

There are infinitely many modal logics.

Review: Models

The **system K** is the set of all K-valid sentences.

The **system T** is the set of all T-valid sentences.

...

Review: Models

- A formal sentence can be *true at* a world in a model. It can't be *valid at* a world.
- A sentence can be *valid in* a class of Kripke models. It can't be *true in* such a class.
- A system of modal logic does not have an accessibility relation.
- A system of modal logic does not have any rules.
- A system of modal logic does not contain any schemas.

Review: Models

(a) Is $\Box\Box p \rightarrow \Box p$ an instance of the **T** schema $\Box A \rightarrow A$? (max 1 word)

Yes.

Review: Models

(b) Is every Kripke model reflexive? (max 1 word)

No.

(c) Explain why every K-valid $\mathfrak{L}_M$-sentence is S4-valid. (max 50 words)

A sentence is K-valid iff it is true at all worlds in all Kripke models.

A sentence is S4-valid iff it is true at all worlds in all Kripke models whose accessibility relation is reflexive and transitive.

If a sentence is valid in all Kripke models, then it is also valid in those Kripke models whose accessibility relation is reflexive and transitive.

(d) Explain why $\Box p \rightarrow p$ is (intuitively) invalid if the box is understood as belief. (max 50 words)

$\Box p \rightarrow p$ is valid iff it is true in every conceivable scenario under any interpretation of p.

If the box is understood as belief, it is easy to conceive of a scenario in which $\Box p \rightarrow p$ is false, on some interpretation of p.

For example: someone believes that it is raining but it is not raining.

Review: Models

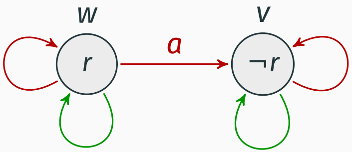
Possible exam questions:

1. Give an example of an $\mathfrak{L}_M$-sentences that is K-valid but that is not an instance of the **K**-schema $\Box(A \to B) \to (\Box A \to \Box B)$.
2. Give an instance of the **T**-schema $\Box A \to A$ that is K-valid.
3. Explain why the **T**-schema $\Box A \to A$ is (intuitively) invalid in deontic logic.
4. Give an example of a set of $\mathfrak{L}_M$-sentences for which there is no class of Kripke models in which all and only the members of this set are valid.

Review: Models

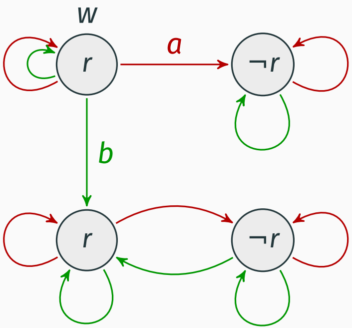
(a) Draw a temporal model in which $\mathsf{F}\,p$ and $\mathsf{F}\,\mathsf{G}\,\neg p$ are both true at some time t.

(b-) Draw a Kripke model for epistemic logic, in which the following is true at *w*: Alice does not know whether it is raining, but Alice knows that Bob knows whether it is raining. (Let *r* translate 'it is raining'.)



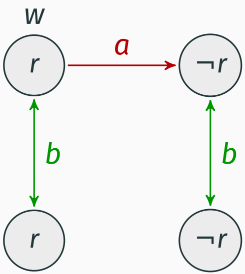
- A model only directly fixes the truth-value of atomic sentences at worlds!
- Epistemic models are reflexive!
- Multi-modal epistemic models contain different kinds of arrows!

(b) Draw another Kripke model, in which the following is true at w: Alice does not know whether it is raining; Alice knows that Bob knows whether it is raining; but Bob does not know that Alice knows that Bob knows whether it is raining.



(b) Draw another Kripke model, in which the following is true at *w*: Alice does not know whether it is raining; Alice knows that Bob knows whether it is raining; but Bob does not know that Alice knows that Bob knows whether it is raining.

Wrong:

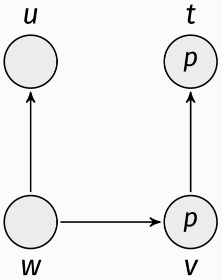


Review: Models

Exam question 2 is a bit like this:

In the Kripke model on the right, p is true at v and t and false at w, and u.

1. Is the model euclidean?
2. At which worlds in the model is $\Box p$ true?
3. For each world in the model, find an $\mathfrak{L}_M$-sentence that is true only at that world.



Review: Trees

Exam question 3:

> For each of the following claims, give either a tree proof or a countermodel.
>
> 1. ...
> 2. ...
> 3. ...

You have to master the K-rules, and how to add further rules.