Logic 2: Modal Logic

Lecture 22

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Review: Frames and correspondence

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A Kripke model has three parts: W, R, V.

When we define validity (or entailment) in terms of a class of Kripke models, we never put constraints on V.

- A is **T**-valid iff A is valid in the class of Kripke models with a reflexive accessibility relation.
- A is **K′**-valid iff A is valid in the class of Kripke models with finitely many worlds.
- A is **X**-valid iff A is valid in the class of Kripke models in which $V(p, w) = 1$ for all $w \in W$. ???

p is X-valid, but q is not!

Review: Frames and correspondence

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When we define validity (or entailment) in terms of a class of Kripke models, we never put constraints on V.

We effectively define validity with respect to a class of Kripke frames.

A **frame** is a model without an interpretation function.

A sentence is **valid on a frame** iff it is true at all worlds in all models based on that frame.

Review: Frames and correspondence

Constraints on the accessibility relation often **correspond** to modal schemas, in the sense that the schema is valid on a frame iff the frame satisfies the constraint.

	Schema	*Corresponding Frame Condition*
T	$\Box A \to A$	R is reflexive: every world is accessible from itself
D	$\Box A \to \Diamond A$	R is serial: every world can access some world
B	$A \to \Box\Diamond A$	R is symmetric: whenever wRv then vRw
4	$\Box A \to \Box\Box A$	R is transitive: whenever wRv and vRu, then wRu
5	$\Diamond A \to \Box\Diamond A$	R is euclidean: whenever wRv and wRu, then vRu such that vRt and uRt

Review: Soundness and completeness

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For almost every conception of validity (or logics) that we've studied, there are proof methods that allow us to demonstrate that a sentence is valid.

A method is **sound** if anything that's provable with the method is valid.

A method is **complete** if anything that's valid is provable with the method.

Review: Soundness and completeness

For example:

- The K-rules for tree proofs are sound and complete with respect to K-validity.
- The S5-rules for tree proofs are sound and complete with respect to S5-validity.
- The KD45-rules for tree proofs are sound and complete with respect to KD45-validity.
- ...

Review: Extending modal propositional logic

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In **multi-modal logic** we have several boxes and diamonds. In easy cases, these are interpreted by several accessibility relations on the same set of worlds.

- $\Box_1 A$ is true at w iff A is true at all R_1-accessible worlds from w.
- $\Box_2 A$ is true at w iff A is true at all R_2-accessible worlds from w.

Multi-modal logics are commonly used in epistemic logic, if want to talk about what different agents know.

We may also want to reason about the connection between knowledge and belief, knowledge and circumstantial possibility, knowledge and obligation, obligation and circumstantial possibility, obligation and time, ...

Review: Extending modal propositional logic

Sentences or schemas that contain more than one type of box or diamond are called **interaction principles**.

Interaction principles often correspond to joint constraints on different accessibility relations.

Schema	*Corresponding Frame Condition*
$\Box_1 A \rightarrow \Box_2 A$	Whenever wR_2v then wR_1v
$\Diamond_1 A \rightarrow \Box_2 \Diamond_1 A$	Whenever wR_1v and wR_2u then uR_1v.

Review: Extending modal propositional logic

If the accessibility relation is linear, it is natural to have forward-looking and backward-looking operators, as in temporal logic.

Branching structures can be conceptualized as divergent linear histories.

On this “Ockhamist” conception, forward-looking and backward-looking operators can be combined with operators that quantify over histories.

Review: Extending modal propositional logic

In Ockhamist semantics, truth is defined relative to an extra parameter that does not reflect an aspect of a conceivable scenario: a history.

We also need an extra (time or world) parameter if we want a ‘Now’ operator (or the modal analogue, ‘Actually’).

In first-order logic, we have an extra parameter g for assignment functions.

The method of supervaluation can be used to reduce extra parameters.

$M, t \models A$ iff $M, t, h \models A$ for all histories h through t.

Review: Extending modal propositional logic

Operators for obligation and permission arguably quantify over the best of the circumstantially accessible worlds.

To make this explicit, models of deontic logic need

- a circumstantial accessibility relation, and
- an ordering of worlds as better and worse.

Review: Extending modal propositional logic

With deontic ordering models, we can also define a conditional obligation operator:

$\mathrm{O}(p/q)$ is true at w iff p is true at the best of the circumstantially accessible worlds at which q is true.

Formally, $A \mathbin{\Box\!\!\rightarrow} B$ has the same interpretation. The ordering is interpreted not as "better relative to w" but as "more similar to w".

Review: Extending modal propositional logic

Some think subjunctive conditionals in English should be formalized in terms of $\boxright$.

Others think they are strict conditionals $A \strictif B$.

Some think indicative conditionals in English should be formalized in terms of $\boxright$.

Others think they are strict conditionals $A \strictif B$.

Others think they are material conditionals $A \rightarrow B$.

Others think indicative conditionals do not express propositions at all.

Review: Extending modal propositional logic

Modal predicate logic is a multi-modal logic in which quantifiers of the form $\forall x$ and $\exists x$ range over different things as quantifiers of the form $\Box$ and $\Diamond$.

Two main questions arise in the semantics of modal predicate logic:

- Can the domain of individuals vary from world from world?
- Is the reference of names and variables constant from world to world?

Difficult Comprehension Questions

(a) Show that system K contains no instance of the schema $\Box\Diamond A$.

In the frame below, every instance of $\Box\Diamond A$ is false at w. So no instance of $\Box\Diamond A$ is true at all worlds in all models.

- You can't assume that A is true at v.
- It is not enough to show that $\Box\Diamond A$ is not K-valid. (Compare K-valid instances of $\Box A \to A$.)

Difficult Comprehension Questions

(b) Is $\Box(\Box A \to A) \to (\Diamond\Box A \to \Box A)$ valid in the class of Kripke models in which R is a linear order? If yes, explain briefly. If no, give a counterexample.

For a counterexample, $\Box(\Box A \to A)$ and $\Diamond\Box A$ must be true while $\Box A$ is false.

schema	true at a world w in a linear model iff
$\Box A$	A is true all points "in the future" of w.
$\Box A \to A$	If A is always going to be true, then A is true.
$\Box(\Box A \to A)$	A is true at any future point after which A is always true.
$\Box(\Box A \to A)$	No $\neg A$-point in the future is followed only by A-points.
$\Diamond\Box A$	There is a point in the future after which A is always true.

Difficult Comprehension Questions

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Example:

$W =$ the set of real numbers

wRv iff $w < v$

$V(p, w) = 1$ iff $w \geq 1$

Here, $\Box(\Box p \to p)$ and $\Diamond\Box p$ are true at 0, and $\Box p$ is false.

Difficult Comprehension Questions

NB: $\Box(\Box A \to A)$ is valid on all and only the frames that are shift reflexive. But an instance of $\Box(\Box A \to A)$ can be true at a world in a model even if the model isn't shift reflexive.

(c) Does the canonical model for K contain worlds w and v such that neither wRv nor vRw (and $w \neq v$)? Explain. (max 100 words)

In the canonical model for K,

- W is the set of all maximal K-consistent sets of sentences,
- wRv iff v contains all sentences A for which w contains $\Box A$.

The following two sets are K-consistent: $\{\neg p, \Box q\}$, $\{\neg q, \Box p\}$.

By Lindenbaum's Lemma, $\{\neg p, \Box q\}$ can be extended to a maximal K-consistent set w, and $\{\neg q, \Box p\}$ can be extended to a maximal K-consistent set v.

Since v is K-consistent, it does not contain q. But w contains $\Box q$. So v is not accessible from w.
Since w is K-consistent, it does not contain p. But v contains $\Box p$. So w is not accessible from v.

Difficult Comprehension Questions

(d) Suppose for every world there is an even better world (relative to some system of norms). Give a plausible new semantics for O in deontic ordering models so that the **D**-schema remains valid as long as the circumstantial accessibility relation is serial.

Difficult Comprehension Questions

The old semantics:

$\mathrm{O}\,A$ is true at w iff A is true at all the best of the circumstantially accessible worlds.

$M, w \models \mathrm{O}\,A$ iff $M, v \models A$ for all $v \in Min^{\prec_w}(\{u : wRv\})$.

$Min^{\prec_w}(S) =_{\text{def}} \{v \in S : \neg\exists u(u \prec_w v)\}$.

Problem: If among the (circumstantially) accessible worlds, for every world there is an even better world, then there are no best of the accessible worlds.

$Min^{\prec_w}(\{u : wRv\})$ is the empty set.

Then $M, w \models \mathrm{O}\,A$ for all A, and $M, w \models \mathrm{P}\,A$ for no A.

So the **D**-schema $\mathrm{O}\,A \rightarrow \mathrm{P}\,A$ fails.

Difficult Comprehension Questions

New semantics:

$\mathrm{O}\,A$ is true at w iff for every circumstantially accessible world, as you move towards better worlds, A eventually becomes true and remains true.

Formally:

$M, w \models \mathrm{O}\,A$ iff for all u with wRv there is some v with wRv such that (i) not $u \prec_w v$ and (ii) for all t with wRt, if not $v \prec_w t$ then $M, t \models A$.

If R is serial, $\mathrm{O}\,A \to \mathrm{P}\,A$ is now valid.