Natural deduction proofs for modal logic

Natural deduction proofs try to mirror intuitive ("natural") ways of arguing for a conclusion. For example, if you wanted to show that a conjunction $p \wedge q$ is true, an intuitive approach would be to show that p is true, then show that q is true, and then infer that $p \wedge q$ is true. Since people disagree over what kinds of inference are natural, there are many styles of natural deduction. I will not survey all the possibilities. Instead, I will briefly explain how one particular style of natural deduction – known as the *Kalish-Montague* style – can be extended to modal logic.

Let's say we want to prove $(p \wedge q) \to q$, in classical propositional logic. In a Kalish-Montague proof, we'd start by writing down our goal, like this.

1.	Show $(p \wedge q) \to q$	

A (supposedly) "natural" way to prove a conditional $A \to B$ is to assume the antecedent A and derive the consequent B. We might therefore start a subproof in which we try to derive q from $p \wedge q$.

1.	Show $(p \wedge q) \to q$	
2.	$p \wedge q$	ass cd

The annotation 'ass cd' tells us that we're *assuming* $p \wedge q$ for the purpose of a *conditional derivation*. From $p \wedge q$ we can directly infer q, by the rule of "simplification" (also known as "conjunction elimination").

1.	Show $(p \wedge q) \to q$	
2.	$p \wedge q$	ass cd
3.	q	2, s

Having derived q from $p \wedge q$, we can infer $(p \wedge q) \to q$. So we cross out 'Show' from 'Show $(p \wedge q) \to q$' and close off the subproof by putting it in a box.

1.	~~Show~~ $(p \wedge q) \rightarrow q$		
2.		$p \wedge q$	ass cd
3.		q	2 s
4.			2 3 cd

The empty last line indicates that the box was closed by the rule of conditional derivation applied to lines 2 and 3.

A proof can contain several subproofs, and subsubproofs within subproofs. Different subproofs are isolated from one another: if you've introduced an assumption A in one subproof, you can't draw on A in another subproof, except if the second subproof is embedded in the first. Sentences from a higher-up level may be imported into a subproof, by the rule of "repetition".

You can find a complete description of this proof method, with all its rules, in Terence Parson's *Exposition of Symbolic Logic*, which is freely available at sites.google.com/site/tparsons5555/home/logic-text.

The method is easily extended to a range of modal logics. To reflect the duality of the box and the diamond, we need to add a "modal negation" rule *mn*. It is actually four rules:

mn: $\neg\Box\neg A \therefore \Diamond A$ $\quad$ $\neg\Diamond\neg A \therefore \Box A$ $\quad$ $\neg\Box A \therefore \Diamond\neg A$ $\quad$ $\neg\Diamond A \therefore \Box\neg A$

The three dots '$\therefore$' indicate that any instance of the schema on the right can be inferred from the corresponding instance of the schema on the left. So '$\neg\Box\neg A \therefore \Diamond A$' states that one may infer, say, $\Diamond(p \rightarrow \Box p)$ from $\neg\Box\neg(p \rightarrow \Box p)$.

We also need a new type of derivation, *sd* (for "strict derivation"), to derive sentence of the form $\Box A$. Strict derivations use a special kind of subproof that starts with no assumption. Intuitively, the subproof takes you to an arbitrary new world that is accessible from a world at which the sentences you have previously proved (or assumed) are true. Your goal is to prove that A holds at this world. If that is done, the subproof can be closed and $\Box A$ has been shown. In this kind of subproof, you are not allowed to import sentences from outside the subproof by the repetition rule. Instead, you have to use a *modal importation rule*.

The basic importation rule, *im*, says that if some boxed sentence $\Box A$ has been established on a higher-up level in a proof, then you may assume the corresponding sentence A inside a strict derivation.

Here is a proof of $(\Box p \wedge \Box q) \to \Box(p \wedge q)$, using these resources.

1.	~~Show~~ $(\Box p \wedge \Box q) \to \Box(p \wedge q)$	
2.	$\Box p \wedge \Box q$	ass cd
3.	$\Box p$	2, s
4.	$\Box q$	2, s
5.	~~Show~~ $\Box(p \wedge q)$	
6.	p	3, im
7.	q	4, im
8.	$p \wedge q$	6, 7, adj
9.		8, sd
10.		2, 5, cd

On line 6, the modal importation rule *im* is used to import assumption p, based on assumption $\Box p$ on line 3 (which is on a higher-up level in the proof). Similarly for q on line 7. Line 9 indicates that since $p \wedge q$ could be derived for an arbitrary accessible world, we can infer $\Box(p \wedge q)$, by *strict derivation*.

These rules suffice to prove every K-valid sentence. For stronger systems of modal logic, we need further rules.

For example, for the system T we would add the rule

$$\textit{ni:} \quad \Box A \therefore A.$$

For system D, we would instead add

$$\textit{bd:} \quad \Box A \therefore \Diamond A.$$

For K4, we need another modal importation rule. This rule, *im4*, allows you to import sentences of type $\Box A$ unchanged into a strict derivation. The rule is used in the following proof of $\Box p \to \Box\Box p$.

1.	~~Show~~ $\Box p \rightarrow \Box\Box p$	
2.	$\Box p$	ass cd
3.	~~Show~~ $\Box\Box p$	
4.	$\Box p$	2, im4
5.		4, sd
6.		2, 5, cd

K5 requires a similar modal repetition rule, *im5*. This one allows you to import sentences of type $\Diamond A$ unchanged into strict derivations.

If both *ni* and *im4* are added to the natural deduction rules for K, we get a natural deduction system for S4. *ni* and *im5* together yield a natural deduction system for S5. For S4.2, yet another rule, *img*, is needed, which allows importing sentences of type $\Diamond\Box A$ unchanged into strict derivations.