Natural deduction proofs for modal logic

Natural deduction proofs try to mirror intuitive ("natural") ways of arguing for a conclusion. For example, if you wanted to show that a conjunction \( p \land q \) is true, an intuitive approach would be to show that \( p \) is true, then show that \( q \) is true, and then infer that \( p \land q \) is true. Since people disagree over what kinds of inference are natural, there are many styles of natural deduction. I will not survey all the possibilities. Instead, I will briefly explain how one particular style of natural deduction – known as the Kalish-Montague style – can be extended to modal logic.

Let’s say we want to prove \((p \land q) \rightarrow q\), in classical propositional logic. In a Kalish-Montague proof, we’d start by writing down our goal, like this.

1. Show \((p \land q) \rightarrow q\)

A (supposedly) “natural” way to prove a conditional \( A \rightarrow B \) is to assume the antecedent \( A \) and derive the consequent \( B \). We might therefore start a subproof in which we try to derive \( q \) from \( p \land q \).

1. Show \((p \land q) \rightarrow q\)
2. \( p \land q \quad \text{ass cd} \)

The annotation ‘ass cd’ tells us that we’re assuming \( p \land q \) for the purpose of a conditional derivation. From \( p \land q \) we can directly infer \( q \), by the rule of “simplification” (also known as “conjunction elimination”).

1. Show \((p \land q) \rightarrow q\)
2. \( p \land q \quad \text{ass cd} \)
3. \( q \quad 2, s \)

Having derived \( q \) from \( p \land q \), we can infer \((p \land q) \rightarrow q\). So we cross out ‘Show’ from ‘Show \((p \land q) \rightarrow q\’ and close off the subproof by putting it in a box.
1. **Show** $(p \land q) \rightarrow q$

<table>
<thead>
<tr>
<th>2.</th>
<th>$p \land q$</th>
<th>ass cd</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.</td>
<td>$q$</td>
<td>2 s</td>
</tr>
<tr>
<td>4.</td>
<td>[2 3 \text{ cd}]</td>
<td></td>
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</tbody>
</table>

The empty last line indicates that the box was closed by the rule of conditional derivation applied to lines 2 and 3.

A proof can contain several subproofs, and subsubproofs within subproofs. Different subproofs are isolated from one another: if you’ve introduced an assumption $A$ in one subproof, you can’t draw on $A$ in another subproof, except if the second subproof is embedded in the first. Sentences from a higher-up level may be imported into a subproof, by the rule of “repetition”.

You can find a complete description of this proof method, with all its rules, in Terence Parson’s *Exposition of Symbolic Logic*, which is freely available at sites.google.com/site/tparsons5555/home/logic-text.

The method is easily extended to a range of modal logics. To reflect the duality of the box and the diamond, we need to add a “modal negation” rule $mn$. It is actually four rules:

$$mn: \quad \neg \neg A \vdash \diamond A \quad \neg \diamond \neg A \vdash \square A \quad \neg \square A \vdash \diamond \neg A \quad \neg \diamond A \vdash \square \neg A$$

The three dots ‘\vdash’ indicate that any instance of the schema on the right can be inferred from the corresponding instance of the schema on the left. So ‘$\neg \square \neg A \vdash \diamond A$’ states that one may infer, say, $\diamond (p \rightarrow \square p)$ from $\neg \neg (p \rightarrow \square p)$.

We also need a new type of derivation, $sd$ (for “strict derivation”), to derive sentence of the form $\square A$. Strict derivations use a special kind of subproof that starts with no assumption. Intuitively, the subproof takes you to an arbitrary new world that is accessible from a world at which the sentences you have previously proved (or assumed) are true. Your goal is to prove that $A$ holds at this world. If that is done, the subproof can be closed and $\square A$ has been shown. In this kind of subproof, you are not allowed to import sentences from outside the subproof by the repetition rule. Instead, you have to use a modal importation rule.

The basic importation rule, $im$, says that if some boxed sentence $\square A$ has been established on a higher-up level in a proof, then you may assume the corresponding sentence $A$ inside a strict derivation.
Here is a proof of \((\Box p \land \Box q) \rightarrow \Box(p \land q)\), using these resources.

1. **Show** \((\Box p \land \Box q) \rightarrow \Box(p \land q)\)

2. \(\Box p \land \Box q\)  
   **ass cd**

3. \(\Box p\)  
   2, **s**

4. \(\Box q\)  
   2, **s**

5. **Show** \(\Box(p \land q)\)

6. \(p\)  
   3, **im**

7. \(q\)  
   4, **im**

8. \(p \land q\)  
   6, 7, **adj**

9. \(\Box(p \land q)\)  
   8, **sd**

10. \(2, 5, \text{cd}\)

On line 6, the modal importation rule *im* is used to import assumption \(p\), based on assumption \(\Box p\) on line 3 (which is on a higher-up level in the proof). Similarly for \(q\) on line 7. Line 9 indicates that since \(p \land q\) could be derived for an arbitrary accessible world, we can infer \(\Box(p \land q)\), by *strict derivation*.

These rules suffice to prove every K-valid sentence. For stronger systems of modal logic, we need further rules.

For example, for the system T we would add the rule

\[\text{ni:} \quad \Box A \vdash A.\]

For system D, we would instead add

\[\text{bd:} \quad \Box A \vdash \Diamond A.\]

For K4, we need another modal importation rule. This rule, *im4*, allows you to import sentences of type \(\Box A\) unchanged into a strict derivation. The rule is used in the following proof of \(\Box p \rightarrow \Box \Box p\).
1. \textbf{Show} \quad \Box p \rightarrow \Box \Box p \\
2. \quad \Box p \quad \text{ass cd} \\
3. \quad \textbf{Show} \quad \Box \Box p \\
4. \quad \Box p \quad 2, \text{im}4 \\
5. \quad \text{2, sd} \\
6. \quad 2, 5, \text{cd} \\

K5 requires a similar modal repetition rule, \textit{im}5. This one allows you to import sentences of type $\Diamond A$ unchanged into strict derivations.

If both \textit{ni} and \textit{im}4 are added to the natural deduction rules for K, we get a natural deduction system for S4. \textit{ni} and \textit{im}5 together yield a natural deduction system for S5. For S4.2, yet another rule, \textit{img}, is needed, which allows importing sentences of type $\Diamond \Box A$ unchanged into strict derivations.