Logic, Computability and Incompleteness

Incompleteness

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Recap

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Mathematics in the 19th century

By the 1800s, mathematics had become sophisticated, but its foundations were recognized to be shaky.

Cauchy, Weierstrass, Dedekind, Cantor, and others tried to remedy this situation by offering precise definitions and rigorous proofs.

Vision: All mathematical truths should be logically derivable from clearly stated principles (axioms).

Recap

Peano Arithmetic (PA)

$$
\begin{array}{ll}
\textit{PA1} & \forall x \forall y\,(s(x)=s(y) \;\rightarrow\; x=y) \\
\textit{PA2} & \forall x\,(\neg(s(x)=0)) \\
\textit{PA3} & \forall x\,(x+0=x) \\
\textit{PA4} & \forall x \forall y\,(x+s(y)=s(x+y)) \\
\textit{PA5} & \forall x\,(x\times 0=0) \\
\textit{PA6} & \forall x \forall y\,(x\times s(y)=(x\times y)+x) \\
\textit{PA7} & A(0) \;\wedge\; \forall x(A(x) \rightarrow A(s(x))) \rightarrow \forall x A(x)
\end{array}
$$

We'd like to prove that PA is

- consistent: $\nvdash \bot$;
- complete: for every sentence A, either $\vdash A$ or $\vdash \neg A$.

Recap

“Wir müssen wissen, wir werden wissen!” (David Hilbert, 1930)

We must know, we will know!

Gödel's First Incompleteness Theorem (1931):

No sufficiently powerful and consistent formal theory can prove all truths of its domain.

Arithmetization of syntax

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We can code sentences and proofs as numbers.

These numbers are called Gödel numbers.

Properties of sentences and proofs turn into numerical properties of their Gödel numbers.

Example:

- $x_1 \mapsto 2$, $x_2 \mapsto 4$, $x_3 \mapsto 6$, ...
- Being a variable $\mapsto$ being an even number.

'x is even' is expressible in $\mathfrak{L}_A$, e.g. by $\exists y(x = y + y)$.

Arithmetization of syntax

Theorem 8.5
All recursive functions and relations are definable in $\mathfrak{L}_A$.

Theorem 8.3
All recursive functions and relations are representable in every extension of Q.

Examples:

- $\mathrm{Sent}(x)$ is defined by some formula $\textsc{sent}(x)$
- $\mathrm{Prf}_{\mathrm{PA}}(x, y)$ is defined by some formula $\textsc{prf}_{\mathrm{PA}}(x, y)$.
- $x * y$ is defined by some formula $\textsc{concat}(x, y, z)$.

(All these formulas are Σ_1 formulas. They are equivalent to formulas of the form $\exists x_1 \ldots \exists x_n \, t_1 = t_2$.)

Semantic incompleteness

Semantic incompleteness

$\mathrm{Prf}_{\mathrm{PA}}(n, m)$ says that n codes a PA-proof of the sentence with Gödel number m.

$\textsc{prf}_{\mathrm{PA}}(x, y)$ defines $\mathrm{Prf}_{\mathrm{PA}}(x, y)$.

So $\textsc{prf}_{\mathrm{PA}}(\overline{n}, \overline{m})$ is true iff n codes a PA-proof of the sentence with Gödel number m.

Define $\textsc{prov}_{\mathrm{PA}}(x)$ as $\exists x\, \textsc{prf}_{\mathrm{PA}}(y, x)$.

So $\textsc{prov}_{\mathrm{PA}}(\ulcorner A \urcorner)$ is true iff there is a PA-proof of A.

Semantic incompleteness

The Diagonal Lemma

For every PA-formula $A(x)$, there is a sentence G that is equivalent to $A(\ulcorner G \urcorner)$.

Apply the diagonal lemma to $\neg\mathrm{PROV}_{\mathrm{PA}}(x)$.

This gives us a sentence G that is equivalent to $\neg\mathrm{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$.

G is true iff G is not provable in PA.

Semantic incompleteness

G is true iff G is not provable in PA.

Suppose G is provable in PA. Then G is false. So PA is can prove a false statement about numbers. But all axioms of PA are true. So G isn't provable in PA.

So G is true. So there is a true statement about numbers that is not provable in PA.

Tarski’s Theorem

Tarski's Theorem

If T is a consistent extension of Q then membership in T is not representable in T.

1. $T(x)$ represents membership in T if
   - if $\vdash_T A$ then $\vdash_T T(\ulcorner A \urcorner)$;
   - if $\nvdash_T A$ then $\vdash_T \neg T(\ulcorner A \urcorner)$.
2. By the Diagonal Lemma, there is a sentence G such that $\vdash_T G \leftrightarrow \neg T(\ulcorner G \urcorner)$.
3. If $\vdash_T G$ then $\vdash_T T(\ulcorner G \urcorner)$ and $\vdash_T \neg T(\ulcorner G \urcorner)$.
4. If $\nvdash_T G$ then $\vdash_T \neg T(\ulcorner G \urcorner)$ and so $\vdash_T G$.

Tarski's Theorem

Church's Theorem

First-order validity is undecidable.

1. Suppose there was an algorithm to decide validity.
2. Then we could decide whether $\vdash_Q A$ by testing $\models \hat{Q} \rightarrow A$.
3. So membership in Q would be computable.
4. So it would be representable in Q.
5. By Tarski's Theorem, this is impossible.

Gödel's First Incompleteness Theorem

Every consistent, axiomatizable extension of Q is incomplete.

1. Let T be a consistent, axiomatizable extension of Q.
2. If T were complete, we could decide $\vdash_T A$ by simultaneously searching for a proof of A and a proof of $\neg A$.
3. So membership in T would be computable.
4. So it would be representable in T.
5. By Tarski's Theorem, this is impossible.