

Emperors, dragons, and other mathematicalia

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Abstract. One of the most intriguing applications of possibilism is the reduction of mathematical truths. I argue that this is not only technically feasible, but also supported by general methodological considerations, that it reflects a natural understanding of mathematical statements and that it solves most of the philosophical puzzles surrounding mathematics.¹

1 Philosophical puzzles about mathematics

Mathematics is puzzling. What are mathematical theories about? What are numbers, functions, sets and lattices? Where are they? What is their nature? Do they have properties beyond those mentioned in mathematics? Do they really exist?

These are ontological questions. Others are ‘ideological’, in Quine’s sense of the term. Consider “member” (or “element”) in set theory. There is no official definition of this notion. Does that mean we have to rely on supernatural revelation to grasp its meaning? Textbooks often say, somewhere on their first pages, that a set is several things considered as one – something like a flock of sheep –, and that the ‘members’ are simply the original things so considered. But soon we are told that there is an empty set, a unit set of the empty set, a unit set of that set, and so on, none of which can in any sense be called “several things considered as one”. So what, really, do “set” and “member” mean? (cf. [Lewis 1991: §2])

Then there are epistemological worries. Mathematical theories do not appear to describe parts or features of reality to which our senses provide access: numbers can neither be seen nor heard. Unlike microphysical particles, they aren’t even causally responsible for observable phenomena. But then how can we know about them? How can we be confident that our mathematical beliefs track what is the case in that isolated mathematical part of reality? (cf. e.g. [Field 1989: §1.4])

¹ The approach I will present resembles proposals by Paul Fitzgerald [1976], David Lewis [1993], Geoffrey Hellman [1996] and Daniel Nolan [2001: ch.7, 2004].

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Another puzzle concerns the modal status of mathematics. Mathematical truths are generally assumed to be necessary. But why couldn't things in that mathematical part of reality be different? If there are many objects of a certain kind, why couldn't there be fewer of them, or none at all? On the face of it, there is no contradiction in assuming that numbers don't exist, so how come there is no corresponding possibility? (cf. [Rosen 2002])

These puzzles could be easily solved by rejecting the relevant mathematical claims. If it's not true that there are infinitely many primes, nor that there are any numbers at all, we need not worry about their nature and necessity and relation to us. I will not consider this possibility. The aim is to answer the philosophical puzzles – as well as possible – without rejecting mathematics.

2 Learning from logicism

Let's begin with the epistemological problem: how could we explain the correlation between our mathematical beliefs and the mathematical facts? A good step towards an answer is to notice that many mathematical truths can be *proved*. It's not a primitive fact that there are infinitely many primes, a fact we have to grasp in some inexplicable manner, as it were by direct insight into mathematical reality. No, that there are infinitely many primes can be proved from simpler facts, and that is most likely how we come to know it.

The ideological problems can be tackled in a similar way by defining certain mathematical notions in terms of others. For instance, “prime number” need not be accepted as primitive if it can be defined in terms of “division” and “1”.

Taken to its extreme, this idea leads to full-blooded logicism, the project of defining all mathematical notions in purely logical vocabulary, and then showing how all mathematical truths can be proved from purely logical truths. If this works, there will at most remain a few concerns about the status and epistemology of logic, but these are clearly tame in comparison with the problems raised by mathematics.

Unfortunately, full-blooded logicism does not work. For one, we know since Gödel that not all mathematical truths are formally derivable from any neat set of axioms, let alone from no axioms at all. We have to rest content with a neat set of axioms, like the ZFC axioms for set theory or the first-order Peano Axioms for arithmetic, from which at least most interesting facts can be derived. (We may find a complete set of axioms in some higher-order logic, like the second-order Peano Axioms. But this isn't much of an improvement: on any reasonable conception of “proof”, many arithmetical truths will remain unprovable.)

There is a more obvious reason why full-blooded logicism can't work: whatever follows from a logical truth is itself a logical truth. By definition, logical truths are true in every model. But mathematical theories like set theory and arithmetic are false in every finite model. So they can't be derivable from logical truths. Again, the problem generalizes: any axioms for set theory or arithmetic must entail the existence of infinitely many objects, and so cannot be as ontologically innocent as we might like.

Or can they? Perhaps arithmetic does not really presuppose the existence of infinitely many objects. Of course it contains sentences like “there are infinitely many primes”. But perhaps on the correct interpretation, such sentences don’t really commit us to the existence of many primes, just as, arguably, accepting “there is a lack of wine” does not commit us to the existence of a peculiar lack-of-wine entity. That there is a lack of wine only means that there is not enough wine. So perhaps “there are infinitely many primes” only means, say, that it follows from the Peano Axioms that there are infinitely many primes, or that if there were a structure exemplifying the Peano Axioms, it would contain infinitely many things occupying the position of primes.

I am not very enthusiastic about this idea. As I said, the aim is to solve the philosophical problems without rejecting substantial mathematical truths. If there are only finitely many things overall, there can’t be infinitely many primes. And if there aren’t infinitely many primes, Euclid’s Theorem is false. Saying that the sentence “there are infinitely many primes” is nevertheless true because it doesn’t really mean that there are infinitely many primes sounds like cheating. No faithful interpretation of mathematics can get rid of its commitment to lots of objects.

But we can try to hide the commitment, for instance behind ‘abstraction principles’ like ‘Hume’s Principle’:

HP) The number of F s equals the number of G s iff the F s correspond one to one with the G s.

It does sound trivial: how could one deny that if the knives on a table are in one-one correspondence with the forks, then the number of knives is the same as the number of forks? It seems that anyone who rejects (HP) must have misunderstood what we mean by “number” or “corresponding one to one”.

In fact, “corresponding one to one” can be defined in second-order logic: the F s correspond one to one with the G s iff $\exists R(\forall x(Fx \rightarrow \exists!y(Gy \wedge Rxy)) \wedge \forall x(Gx \rightarrow \exists!y(Fy \wedge Ryx)))$. So if we accept second-order logic as logic, the only non-logical term in (HP) is “the number of”. Proponents of the so-called ‘Neo-Fregean’ St. Andrews school thus suggest that (HP) should be regarded as *implicitly defining* “the number of”. That’s why rejecting (HP) is a sign of misunderstanding: one in effect rejects a definition.

Now surprisingly, the full second-order Peano Axioms are derivable from (HP) together with fairly natural definitions of “successor” and “zero”. (The proof is due to Frege; see e.g. [Boolos 1996] for details.) So if second-order logic counts as logic and (HP) as a definition, the second-order Peano Axioms are after all provable from logic and definitions alone.

Let me set aside the status of second-order logic until later and concentrate on (HP). Since the Peano Axioms entail that there are infinitely many things, and (HP) entails the Peano Axioms, (HP) itself, trivial though it may seem, entails that there are infinitely many things.² Could a mere definition, an analytical truth, do that?

² To see how, take the predicate $x \neq x$. There evidently is a one-one correspondence between the (zero) things that fall under this predicate and the things that fall under this predicate. So taking (HP) from right to left, the number of non-self-identical things equals the number of non-self-identical

Of course not. (HP) contains a substantial metaphysical assumption. But none the worse for that. There is no way to get arithmetic without substantial metaphysical assumptions, in particular about the size of reality. The question is whether (HP) is better apt to solve the puzzles raised by arithmetic than, say, the Peano Axioms themselves. But that isn't clear. How, for instance, do we know that (HP) – this trivially sounding substantial assumption – is true? How can we explain its necessity? How does it help us answering questions about the nature and whereabouts of numbers? (HP) doesn't tell us what kind of internal structure numbers have, in what relations they stand to us, and, notoriously, whether Julius Caesar is one of them.

At this stage, Neo-Fregeans usually take rescue in some kind of anti-realist metaphysics: whether an existence claim is correct, they say, is determined by the rules that govern those claims. (HP) is a rule of this kind, allowing us to move from “the *F*s correspond one to one with the *G*s” to “the number of *F*s = the number of *G*s”, and thus to “there is something which is the number of *F*s”. So all that's needed for the truth of the existence claim is that the original correspondence claim was correct. There is no need for a further matching with what there *really* is, objectively, out there. Moreover, since objects introduced by a particular abstraction principle like (HP) constitute a genuine ontological category, they are, according to Neo-Fregeans, never identical with objects belonging to other categories, like Julius Caesar (see e.g. [Hale and Wright 2001]).

Unfortunately, I fail to see how we could make it true that there are infinitely many things of a special ontological category just by adopting a linguistic rule. I could understand if “the number of *F*s = the number of *G*s” should be regarded as a mere *façon de parler*, a slightly misleading way of saying that the *F*s correspond one to one with the *G*s. But then it would be *wrong* to infer that there is at least one number from this *façon de parler*, just as it is wrong to infer that there is at least one lack from “there is a lack of wine” if this is just an odd way of saying that there is not enough wine.

In fact, I don't think it is a good idea to claim, for whatever reason, that the numbers comprise a special ontological category. For this is just what creates all the philosophical puzzles. If it turns out to be the only acceptable interpretation of arithmetic – if there is no way to locate the numbers among more familiar objects –, I'm willing to accept that. But we shouldn't give up without trying.

3 The Canberra Plan

We have what Frank Jackson [1998: ch.1] calls a *location problem*, a problem of finding entities given in a certain vocabulary among entities described in a different vocabulary. Suppose for example that the world can be completely described in some vocabulary not containing “Mount Everest”. Perhaps microphysical vocabulary suffices, or perhaps

things. So there is a number of non-self-identical things. Let's call it 0. This is our first object. Next, consider the predicate $x = 0$. Again the things falling under this predicate correspond one to one with the things falling under it, so just as before we get from (HP) a number – 1 – of things falling under this predicate. Since 0 falls under $x = 0$ and not under $x \neq x$, there is *no* one-one correspondence between these two predicates. Taking (HP) from left to right it follows that $0 \neq 1$, giving us a second object. And so on.

phenomenal and indexical terms must also be used, or perhaps all kinds of macrophysical facts need to be included. In any case, the question arises how these facts make true facts about Mount Everest and which, if any, of the entities mentioned in the basic description is Mount Everest.

A general strategy for solving location problems is the *Canberra Plan*: to locate a certain thing (or property) A among the things (and properties) given in some restricted vocabulary, first collect information we possess about A ; then look for something B in the basic description that satisfies these conditions. The idea is that if A is known to do this and that, and B (and only B) in fact does this and that, then A can be identified with B . In this way, we can presumably identify Mount Everest with some huge network of microphysical particles, water with H_2O , heat with molecular energy, and events with triples of things, times and properties (or sets of spacetime regions, or whatever events are on your favourite account).

None of these identities are immediately obvious or somehow bring to the surface what we ‘always really meant’ by “heat”, “water”, “event” and “Mount Everest”. Off-hand, one might even have thought that Mount Everest is *not* a huge network of little particles with lots of empty space in between. But it would be unwise to dogmatically hold on to this opinion and conclude in the light of scientific evidence that either Mount Everest does not exist at all or is some further entity, miraculously co-located with an aggregate of particles. Likewise, for the benefit of a manageable ontology, we shouldn’t treat events as irreducible *sui generis* entities merely because, intuitively, events aren’t sets.

It is also by applying the Canberra Plan that mathematicians have managed to greatly reduce their primitive ideology. For example, the main feature of ordered pairs – the only feature relevant for mathematical purposes – is that $\langle x, y \rangle = \langle v, w \rangle$ iff $x = v$ and $y = w$. As sets of the form $\{x, \{x, y\}\}$ satisfy this job description, ordered pairs are commonly identified with such sets. Likewise, as the natural numbers can be characterized by the (second-order) Peano Axioms, and both the finite von Neumann ordinals (the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots$) and the finite Zermelo ordinals (the sets $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots$) satisfy these axioms, mathematicians commonly identify the numbers with those ordinals.

However, the numbers can hardly be identical with *both* the von Neumann *and* the Zermelo ordinals, otherwise 2 would be both $\{\{\emptyset\}, \emptyset\}$ and $\{\{\emptyset\}\}$, contradicting the fact that $\{\{\emptyset\}, \emptyset\} \neq \{\{\emptyset\}\}$. In general, if A is characterized as being such-and-such, and several things satisfy the job description, how can we single out the one with which to identify A ?

This happens not only with numbers, but also with ordered pairs, and with mass and Mount Everest: when we speak of “mass”, do we mean rest mass or of relativistic mass? Having agreed to identify Mount Everest with an aggregate of physical particles, exactly which aggregate shall we choose? Lots of aggregates seem equally well suited, depending on where exactly the mountain’s boundaries are drawn.

Some conclude that Mount Everest cannot be an aggregate of particles after all, but instead belongs to the special ontological category of ‘vague objects’ (see [Rosen and Smith 2004]). Similarly, Paul Benacerraf [1965] concludes that the natural numbers cannot be sets. As for mass, this response would mean that we speak of neither rest mass nor relativistic mass (nor of nothing at all), but of a third kind of mass, strangely

ignored by recent physics.

There is something odd about this response: if different things B and C satisfy the job description for A , how does it help to introduce yet another entity D that also satisfies the description? Why identify A with this further entity D rather than B or C ? It seems to me that the introduction of new entities, rather than solving the problem, only makes it worse. An equally bad response, proposed by David Lewis in his early papers on theoretical identification (e.g. [Lewis 1970: 83]), is to conclude that where the conditions are multiply satisfied, the relevant entities – Mount Everest, mass, numbers, etc. – simply do not exist.³

A better response is to conclude that it is simply indeterminate exactly what “Mount Everest”, “mass” etc. denote. One way to make this more precise is supervaluationism: on the supervaluationist interpretation of indeterminacy and vagueness, a sentence containing an indeterminate expression is true iff it is true on all resolutions of the indeterminacy, and false iff it is false on all resolutions. If, like “Mount Everest consists of an even number of atoms”, the sentence is true on some resolutions and false on others, it remains truth-value-less (see [Keefe 2000] for a discussion and defense of supervaluationism).

I don’t want to get into the semantics of vagueness and indeterminacy here. What I want to claim is only that if all we know about some thing A is that it is such-and-such, and if several things turn out to satisfy that condition, then we should conclude that A is F only if *all* the such-and-suchs are F .

4 Locating the numbers

Let’s apply the Canberra Plan to arithmetic. First, we have to gather facts about the natural numbers and their properties. We know, for example, that $2+3$ equals 5 , that there are infinitely many primes, and that all numbers have a unique decomposition into prime factors. All these facts, indeed all arithmetical truths whatever, follow from the second-order Peano Axioms. So an obvious idea is to take the Peano Axioms as characterizing arithmetical things and properties. (Notice that here we do not use the axioms as a foundation from which to derive arithmetical theorems. Rather, we use them to explicate arithmetical terms. The second-order Peano Axioms do this much better than their incomplete first-order counterparts.)

Here is a (slightly uncommon) version of the second-order Peano Axioms, with “successor” as the only primitive:

- 1) Nothing has more than one successor, and nothing is the successor of more than one thing;
- 2) all things that are successors have a successor;

³ Often, when several things satisfy the job description for A equally well, they all satisfy it badly (e.g. because it is part of the job description that nothing else satisfies the relevant conditions). In *this* case it is reasonable to conclude either that A doesn’t exist or that it is some further entity, not identical with any of the considered things. These cases are not cases of multiple satisfaction, but cases of failed satisfaction.

- 3) there is a single thing that has a successor but is not itself a successor;
- 4) whenever there are some things F such that a) the thing mentioned in (3) belongs to the F s, and b) every successor of every thing that belongs to the F s also belongs to the F s, then every thing that has a successor belongs to the F s.

All other arithmetical terms are easily definable in terms of “successor”: *zero* is the unique thing that has a successor but is no successor; x is a *natural number* if x has a successor; x is *less than or equal to* y if y belongs to all things F such that x is one of the F s and any successor of any of the F s is also one of the F s; x is *the number of* F s if the F s correspond one to one with the numbers less than (i.e. less than or equal to but not equal to) x ; and so on.

To complete the application of the Canberra Plan, we only need to find a relation in our basic ontology that satisfies the conditions these axioms lay down for the successor relation.

What shall we take as the basic ontology? If we restrict it to a narrow range of things – say, the books on my shelf –, it will not be very interesting to learn that the numbers cannot be found among them. So we should include everything we believe in, except perhaps Platonic mathematical entities.

Now I suggest that we include not only all things of which we believe that they actually exist, but also all things of which we believe that they might have existed. I apologize if that is *more* than what you believe in.

Things that might have existed – *possibilia* – are things that exist at other possible worlds. Possible worlds correspond to ways the world might be. There could have been dragons, and there could have been a planet made entirely of chocolate; so there is a possible world containing a dragon, and a world containing a chocolate planet. For present purposes, the metaphysics of merely possible objects is not very important. The merely possible dragons might be concrete and physical animals, occupying their own spatiotemporal universe; or they might be some kind of representations, or *sui generis* modal entities. What matters here is only that they exist, and that they are not mathematical entities.

Do merely possible things really exist? The fact that we appear to talk about them when we describe what might have been the case is hardly conclusive. But *possibilia* also prove of great use in more sophisticated theorizing, in intensional semantics and probability theory, and in many parts of philosophy (see [Lewis 1986: ch.1]). This is a good, but not decisive, reason to believe in them. I will offer you another reason: that if you accept *possibilia*, you can eschew the Platonic realm of *mathematicalia* and solve the puzzles it raises.⁴

⁴ Some have argued that talk about alternative possibilities should not be understood literally, but paraphrased away, like talk about lacks. I don’t want to preclude this possibility, though the currently available suggestions, e.g. [Armstrong 1989] and [Rosen 1990], don’t look very promising to me. Even if their technical problems could be overcome, they would deliver strange results: if modal and counterfactual and mathematical statements are to be interpreted in terms of *possibilia*, and statements about *possibilia* in terms of, say, what is true according to a certain fiction, modal and

So assume possibilia are in our basic ontology. The next question is whether there is a relation on the things in that ontology which satisfies what the Peano Axioms say about the successor relation. On a sparse conception of relations (on which a relation is something like a natural, polyadic universal), this is far from obvious. On an abundant conception however, on which there is a two-place relation for every class of ordered pairs, it is quite plausible. Indeed, it is not difficult to see that such a relation exists if only (and only if) there are infinitely many things in our ontology.⁵

I think it is safe to assume that there are infinitely many possibilia. Hence there is a successor relation satisfying the Peano Axioms. Or rather, *there are* such relations: any relation that results from a successor relation by permutation of its relata is also a successor relation. That is, if one successor relation identifies me with the number 6 and Julius Caesar with the number 7, the relation that is just like this one except that Caesar and I have traded places is also a successor relation.

Thus we have a (rather extreme) case of indeterminacy: lots of relations in our basic ontology perfectly satisfy the successor conditions. In line with what I said in the previous section, we should therefore regard an arithmetical sentence as true iff it is true for *all* relations on possibilia satisfying the successor conditions expressed by the Peano Axioms.

5 Possibilist (eliminative) structuralism

The interpretation reached in the previous section closely resembles *eliminative structuralism* in the philosophy of mathematics. (Non-eliminative structuralism is the view that the object of mathematical studies are abstract entities called “structures”, see e.g. [Parsons 1990]. From now on I will mean eliminative structuralism by “structuralism”.)

Structuralism was developed by Frege [1906] in an attempt to understand some remarks of Hilbert and his allies on the foundations of geometry. Hilbert had claimed that his axioms of geometry should be regarded as *defining* the terms “point”, “straight line”, “between” etc. they contain. Frege found that unacceptable, for largely the same reasons for which I’ve rejected the Neo-Fregean approach: Hilbert’s ‘definitions’ don’t settle whether (or under what conditions) Julius Caesar is a point, and they entail the existence of infinitely many objects, which mere definitions can’t do. Frege suggested that perhaps on Hilbert’s view, “point”, “line” etc. aren’t really terms with a fixed meaning – somehow defined by the axioms –, but rather universally bound variables,

counterfactual and mathematical statements are in effect interpreted as statements about what is true in a certain fiction, which, as I mentioned in section 2, I find hard to believe. Nevertheless, if some reinterpretation programme of talk about possibilia proves feasible, what I’m going to say will not be undermined. It will simply have to be reinterpreted accordingly.

⁵ Consider the ‘Ramsey sentence’ $\exists R \text{ PA}_2(\text{successor}/R)$ of the Peano Axioms, where all occurrences of “successor” in PA_2 have been replaced by the existentially bound variable R . This sentence contains only logical vocabulary, hence whether it is true in a given model can only depend on the size of the domain, not on the assignment to non-logical constants. As PA_2 is true in some countably infinite model, and remains true when further elements are added to the domain (without changing the interpretation of “successor”), it follows that its Ramsey sentence is true in all infinite models.

restricted by the axioms. (Though Frege preferred to reserve “axiom” for basic assumptions on which mathematical theories are built, as opposed to conditions explicating mathematical notions.) On this reading, geometrical statements do not describe special geometrical objects. They are abbreviate quantified conditionals: “whenever some property *Point*, relation *Between*, etc., together satisfy such-and-such conditions (the axioms), then ...” (see in particular [Frege 1906: 304–309]).

Which is more or less just what I suggested in the previous section concerning arithmetic. A minor difference concerns truth-values other than *true*: on Frege’s structuralist interpretation, an arithmetical sentence is false iff it is false on *some* assignment of values to “successor” satisfying the Peano Axioms, whereas on the supervaluationist reading it is false iff it is false on *all* such assignments. Another difference is that the structuralist interpretation is often understood as a claim about the ‘deep structure’ or ‘logical form’ of sentences: arithmetical sentences are somehow supposed to really *be* abbreviated universal conditionals. This understanding is less widespread for the supervaluationist conditionals. At any rate, I think it is better to leave speculations about these matters to linguists.⁶

Structuralism is very natural at least for some parts of mathematics: when we learn in group theory that there is a unique identity element, we presumably don’t learn that there is a unique object in mathematical heaven, *the* identity element. What we learn is rather that any things satisfying the group axioms contain a unique thing satisfying the conditions for the identity element.

The main problem for structuralism is the antecedent of the universal conditionals: how can we be sure that the axioms of group theory or geometry or arithmetic are satisfied? If not, the quantifiers in the structuralist analysis are empty, and all sentences belonging to the relevant theory get assigned the same truth value (true on Fregean structuralism, false on the indeterminacy interpretation, which here again looks more natural). And even if we are sure that the axioms are actually satisfied, how can we be sure that they are *necessarily* satisfied, as we should be if we want to uphold the necessity of mathematical truths?

These worries disappear if we read the structuralist quantifiers as possibilist quantifiers. It is uncertain and contingent whether any relation on actually existing things satisfies the successor conditions expressed by the Peano Axioms – because it is uncertain and contingent whether there are infinitely many things. But we may confidently assume that there are, and necessarily are, infinitely many possible things.

When Frege asked Hilbert how he could be sure that his axioms of geometry are satisfied, Hilbert replied:

If arbitrarily selected axioms together with all their consequences do not contradict one another, then they are true, and the things defined by the

⁶ In cases like geometry, where several primitives occur in the axioms, resolutions of the indeterminacy are simultaneous assignments of values to “point”, “line”, “between” etc.: if on some resolution, points are *A*-things and lines *B*-things, while on another points are *C*-things and lines *D*-things, there may be no admissible resolution on which points are *A*-things and lines *D*-things (e.g., let $A = D$). Such interconnected indeterminacies are often missing in presentations of supervaluationism.

axioms exist. For me, that is the criterion of truth and existence.⁷

As a principle about actual existence, this is certainly crazy (which is what Frege thought about it). But as a principle about *possible* existence, it makes evident sense: if some conditions do not contradict one another, then it is indeed possible that something satisfies the conditions. One might say this is just what “not contradicting one another” means.⁸

There is something funny about the antecedent worry anyway: mathematicians do not and need not care whether any actual objects happen to exemplify certain mathematical conditions (say, the conditions for a Hilbert space). It suffices if the conditions *could* be satisfied. This is why consistency, rather than correspondence with actuality, is what matters in mathematics.

From this point of view, using possibilia in the analysis of mathematical statements is not just a technical trick to ensure that the antecedents are satisfied. On the contrary, it reflects a very natural attitude towards mathematics.

It also explains another puzzling feature, noticed for example by Burgess and Rosen [2005: §2]: neither mathematicians nor physicists seem to be worried about appealing to vast realms of mathematical entities in their theories. In mathematics, the guiding principle seems to be maximizing rather than minimizing ontology. If mathematics deals with the space of possibility, this is only to be expected: in modal affairs, existence claims are generally unproblematic; it’s *non*-existence claims that need defense and explanation.

Ignoring the minor differences between the interpretation sketched in the previous section and the kind of Fregean structuralism just outlined, I will use “possibilist structuralism” for the general idea they share.

6 Properties, relations, fusions and plurals

So far, I have taken for granted second-order logic and thereby the existence of abundant properties and relations, assuming that for any set or set of pairs of things in our ontology there is a corresponding property or relation, respectively. One might wonder whether these alleged properties and relations are anything else but those sets, in which case possibilist structuralism would presuppose the existence of sets. So it may be worthwhile to see if we can do without them.

A nice way to achieve this at least for properties is provided by George Boolos’s [1984, 1985] plural interpretation of second-order quantification. On this interpretation, second-order quantifiers don’t range over special entities, but rather correspond to certain plural constructions common in natural language: Whereas on the traditional interpretation, $\exists F Fa$ is read as “there is a set such that a is a member of it”, Boolos reads it as “there are some things such that a is one of them”. Similarly, $\exists F \forall x (Fx \leftrightarrow x = a \vee x = b)$, can be understood plurally as “there are some things such that for all x , x is one of them iff

⁷Hilbert, letter to Frege, 29 December 1899, [Frege 1976: 66], my translation

⁸I don’t want to speculate about what Hilbert meant by his remark. Note that it predates both Hilbert’s programme and the modern distinction between formal consistency and satisfiability.

either x is a or x is b ". This does not commit us to the existence of any set or class or collection containing a and b . So instead of talking about properties, I could have used plurals, as in fact I sometimes have.

However, plural quantification can only replace *monadic* second-order logic. Further devices are needed to replace talk about (two-place) relations. Such devices have been developed by John Burgess and Allen Hazen in the appendix to [Lewis 1991], where they show that plural quantification together with classical, extensional mereology can go proxy for monadic *third-order* logic.

Mereology is the theory of parts and wholes. Usually, "part" is taken as primitive in mereology. An *atom* is then defined as something that does not have proper parts; a *fusion* of some things F is defined as a thing that has all the F s as parts and no part not overlapping any of the F s. (Some things *overlap* if they share a part.) Classical, extensional mereology mainly consists of the following principle, which I will call the *Fundamental Principle of Mereology*:

FPM) For any things there exists exactly one fusion of those things.

The basic idea of Burgess and Hazen is now to replace quantification over relations by plural quantification over pairs, where pairs $\langle x, y \rangle$ are not defined set-theoretically, but mereologically (with the help of some further plural quantification). The details of this mereological definition of pairhood are intricate, and not directly relevant here, so the interested reader is directed to the appendix of [Lewis 1991], and [Lewis 1993: 222-224].

Using these tools, whatever I said about relations can be rephrased in terms of plurals and mereological ersatz pairs. For instance, when I said that some relation on possibilia satisfies the conditions expressed by the Peano Axioms, I could instead have said that there *are* mereological ersatz pairs R of possibilia that jointly satisfy those conditions. (That the R s jointly satisfy the condition that nothing has more than one successor, for example, means that whenever ersatz pairs $\langle x, y \rangle$ and $\langle x, z \rangle$ are among the R s, then $y = z$.) The existence of such pairs is guaranteed (given the satisfiability of the Peano Axioms) by the unrestrictedness of plural comprehension and mereological composition.

7 Locating the sets

How does possibilist structuralism carry over to set theory?

We can proceed as before: First, follow the set theorists in defining all set theoretical notions in terms of "membership"; then choose, say, the ZFC axioms as providing the job description for membership 'relations' (that is, for mereological ersatz pairs $\langle x, y \rangle$ such that x is a member of y ⁹). Finally, show that the job description is satisfied by 'relations' on possibilia. A set theoretical sentence should be understood as true iff it is true for all these 'relations'. Apart from invoking possibilia and the plural-mereological substitute for relations, this is the approach suggested by Paul Fitzgerald [1976].

⁹This time, there is no set of ordered pairs corresponding to the 'relation'. Otherwise that set, having members, would have to be among the paired entities it contains, contradicting the axiom of Foundation.

In fact, as in the case of arithmetic, we better use *second-order* ZFC. Thereby we can avoid an infinite number of (replacement) axioms in the antecedent of the structuralist conditionals, and, more importantly, we can rule out countable models: it may be indeterminate exactly what we're talking about when we talk about sets, but it is clear that we're not talking about things of which there are only countably many; but the first-order ZFC axioms are satisfied by 'membership relations' on a countable domain.

However, unlike second-order Peano Arithmetic, even second-order ZFC is not categorical, that is, not all its models are isomorphic to one another. It is only 'quasi-categorical', in that whenever there are two models of second-order ZFC, one is isomorphic to an initial segment of the other. Thus we could ensure that set theoretic statements always have the same truth value on all choices of the membership 'relation' by formulating the axioms in such a way that absolutely everything counts as a set. This ties the size of the allowed models to the size of reality – to the number of things in our basic ontology –, so that the remaining membership 'relations' can only differ by a permutation of individuals.¹⁰

For this interpretation to work, our basic ontology must again be sufficiently large. In the case of arithmetic, only countably many things were needed, but now we need as many as there are sets in set theory, which is really quite a lot.

Fortunately, there are independent reasons to believe in that many possibilities. Here is one: suppose there is a cardinality κ of all possibilities. Then it is impossible that there be 2^κ objects; but why should this be impossible? After all, "there are 2^κ objects" is not contradictory. Could it be *a posteriori* impossible, like "Hesperus is not Phosphorus"? Hardly. We have an *explanation* why "Hesperus is not Phosphorus" is impossible (based on a certain theory of names), but this explanation does not work for "there are 2^κ objects". More generally, that Hesperus is necessarily Phosphorus can be found out by empirical investigations. (That's why it's *a posteriori*.) But it is quite unintelligible how empirical investigations could show us that there are necessarily no more than κ objects. This kind of necessity would be a strange and unusual sort of necessity, one that is neither analytically false nor discoverable by empirical investigations. It seems to me that there is no reason to believe in such necessities. Indeed, if possibilities are to do their job in the interpretation of mental and linguistic content, 'strong' necessities of this kind should be rejected (cf. [Chalmers 2002: 192–4]); see [Nolan 1996: §4] and [Bricker 1991] for further reasons to believe that there are as many possibilities as sets).

8 Non-formal conditions and Lewisian sets

Up to this point, I have assumed that all we know about numbers and sets are truths of arithmetic and set theory. But perhaps there are other conditions we should add to the job descriptions.

¹⁰ Curiously, the assumption that everything is a ZFC set conflicts with the Fundamental Principle of Mereology (see [Uzquiano 2005]): ZFC models have inaccessible size, but unless there is a lot of atomless gunk, the Fundamental Principle entails that the number of things in reality is accessible. In response, one could either claim that there is enough atomless gunk in logical space, or employ a stronger set theory such as MK that allows for proper classes as well as sets.

An obvious candidate would be the condition that ordinary objects like Julius Caesar are neither numbers nor sets. This sounds intuitively right. However, I have argued above that we should not take intuitions of this kind too seriously, lest we burden ourselves with a needlessly complicated ontology and philosophical puzzles. Remember how bad it would have been to require that events are not sets or mountains not networks of little particles with empty space in between. We should be open for ontological discoveries.

The assumption that Julius Caesar is not a set might seem important for *impure* set theory, whose domain is divided into sets and non-sets, or ‘individuals’. Shouldn’t Caesar be on the ‘individuals’ side of this divide? Not necessarily. The main purpose of impure set theory is to provide sets of ordinary objects. Certainly we *should* require that Julius Caesar is a member of some sets, so that there are sets like the set of Roman emperors. But to this end, Caesar need not be an ‘individual’. He can also be a member if he is a set, or if it is indeterminate whether he is a set or an ‘individual’. Indeed, if, as I suggested in the previous section, we require all things to be sets in our axioms of pure set theory, possibilist structuralism gives us a set of Roman emperors even in *pure* set theory.¹¹

David Lewis, in his ‘megethological’ reconstruction of impure set theory ([Lewis 1991], [Lewis 1993]), adds two non-formal conditions to the job description for sets. One is the condition I have rejected: that ordinary things not be sets ([1993: 221]). The other is that sets have a certain mereological structure.

According to Lewis, there are not two different ways of ‘considering several things as one’: the mereological and the set-theoretical way, delivering, respectively the fusion and the set of the considered things. The only fundamental kind of composition is mereological. Sets are mereological fusions. The parts of a set, however, are not its members but its subsets. Hence the set of primes, for example, is literally part of the set of natural numbers.

On this conception, the atomic constituents of a set are the *unit sets*, or *singletons*, of its members. In general, x is a *member of* set y iff the singleton $\{x\}$ of x is part of y . Membership is thus reducible to the special case that relates things x to their singletons $\{x\}$.

Lewis shows how the axioms of classical set theory are then derivable from a few basic assumptions about this singleton relation, provided only that reality contains enough objects ([1993: 220f., 227f.]). More interestingly, he also shows that if there are enough objects, then there are singleton ‘relations’ that satisfy those assumptions ([1993: 224–6]). This is no longer obvious once the assumptions contain mereological, as opposed to purely logical, vocabulary.

So it may seem that Lewis’s set theory lends itself easily to possibilist structuralism. In fact, Lewis also advocates a structuralist interpretation. His account is not *possibilist*

¹¹ This is because there are only few (actual) Roman emperors. By contrast, neither pure nor any kind of impure set theory can give us a set of *all possibilities*, or all things that are *not* Roman emperors, if these things exceed every cardinality.

In a set theory with proper classes (see fn.10), things get more complicated: if Julius Caesar is a proper class on some choice of the membership ‘relation’, he will not be a member of any set (or class).

structuralism merely because he requires that the sets inhabit a special part of reality. Can this requirement simply be dropped? Unfortunately not. The trouble is caused by some oddities of Lewis's account that I have not yet mentioned.

I said that the atomic constituents of sets are singletons. But if subsets are parts, isn't the empty set \emptyset a proper part of any singleton $\{x\}$? And if so, what is the rest of $\{x\}$, the mereological difference $\{x\} - \emptyset$? Not $\{x\}$ again, otherwise $\{x\}$ would be a *proper* part of itself. Nor could it be the empty set, as the fusion of \emptyset and \emptyset is \emptyset , not $\{x\}$. But $\{x\}$ and \emptyset are the only subsets of $\{x\}$ and hence should be its only parts. Lewis solves this problem by counting only *non-empty* subsets as parts of a set. So not all sets are fusions of singletons: the empty set is not. Lewis instead defines it as the fusion of all objects that do not have singletons as parts. Such objects Lewis calls "individuals". ([Lewis 1993: 210–2])

It immediately follows that on this terminology, some things are neither sets nor individuals. For by the Fundamental Principle of Mereology, there are fusions of singletons and individuals. These *mixed fusions* are neither individuals (they have singletons as parts) nor sets (they are not fusions of singletons alone, nor are they the empty set). Moreover, again by mereology, there is a fusion of *all* singletons. But there is no union set of all sets in standard set theory. In Lewis's set theory, such large fusions of singletons are not sets, but *proper classes*. ([1993: 208f., 212f.])

Lewis stipulates that mixed fusions and proper classes do not have singletons. This is not entirely arbitrary, for there are strictly *more* mixed fusions and proper classes than there are singletons: by mereology, there is no one-one correspondence between the fusions of any atoms and the atoms themselves. Hence there are more fusions of singletons than there are singletons. Since the singleton relation is a one-one map from the sets into the singletons, it follows that there can be no such map for the remaining fusions of singletons, the proper classes. As for every proper class, its fusion with any individual is a mixed fusion, the mixed fusions also outnumber the singletons.

Now consider what happens if we drop Lewis's condition that ordinary things must not be sets. The division between singleton atoms and individual atoms will then vary with different choices of the singleton 'relation'. On some choices, certain atoms in Caesar's brain will count as singletons and others as individuals. But then Caesar is a mixed fusion on that choice of the singleton 'relation', and thus not a member of any set! Yet among the (non-formal) things we know about sets is that some of them have Julius Caesar as a member.

This time we can't simply demand that all things must count as sets (or proper classes) on all admissible singleton 'relations', as we could in structuralism based on ZFC. For we need an empty set, which, being memberless, cannot be a fusion of singletons. So there is no way to rule out mixed fusions of sets and the empty set. (For what it's worth, we could stipulate that there is only one atom that is not a singleton.)

The easiest fix is probably to allow singletons for *some* mixed fusions. Just as Lewis allows singletons for all 'small' fusions of singletons – those that are not proper classes –, so we can allow singletons for all 'small' mixed fusions, more precisely, for those mixed

fusions that do not contain as many atoms as there are sets (cf. [Lewis 1993: 221]).¹² Since all Roman emperors are small in this sense, this will guarantee a set of Roman emperors.

It may nevertheless appear odd to claim that some ('large') fusions of ordinary possibilities are not members of any set. But this can hardly be avoided if the possibilities, and the atoms of possibilities in particular, exceed every cardinality. For then there are at least as many atoms of possibilities as there are singletons, and therefore strictly *more* fusions of these atoms. So they cannot all have a singleton.

Incidentally, this also shows that Lewis's original [1993] framework, in which *all* fusions of objects outside the mathematical realm have a singleton, breaks down if there are proper-class many possibilities – something Lewis only came to accept later (see e.g. [Lewis 2002: 8]).

9 Conclusion

Logicism and its modern descendants fail because the substantial ontological implications of mathematics can't be explained away. If our mathematical theories are true, there must be lots of objects. But we need not assume that these objects inhabit a special part of reality or have an otherwise special nature.

Instead, I have suggested that we interpret mathematical statements as quantifications over possibilities (plus whatever else there is in our ontology). I have argued that this interpretation results from applying a general ontological strategy, the Canberra Plan, and that it also corresponds to a fairly natural attitude towards mathematics. I will close by taking another look at the puzzles with which I began.

The ontological and ideological puzzles are solved. Like the impossible logicians, we need neither a special mathematical ontology nor any primitive mathematical ideology. Undefined mathematical terms are in effect replaced by universally bound variables.

As for the epistemological puzzle, no direct insight or intuition into an otherworldly realm is required. What we need to know in order to know, say, an arithmetical truth is only that a) it follows from the conditions for successor 'relations' expressed by the arithmetical axioms, and b) that the conditions could be satisfied (and therefore *are* satisfied among possibilities).

Finally, the modal status of mathematical truths is no longer a mystery: on the possibilist interpretation, mathematical statements are unrestricted modal statements, and unrestricted modality obeys the S5 principles: an unrestricted modal statement is

¹²Lewis's second condition on singleton relations, that their domain consists of all small fusions of things in their range and all things not overlapping any things in their range, can then be simplified: the domain of a singleton relation simply consists of all small things.

A different solution is offered by Daniel Nolan [2001: ch.7, 2004], who unlike Lewis counts the empty set as a proper part of any set. $\{x\}$ then consists of \emptyset and $\{x\} - \emptyset$, the latter of which Nolan calls the "Esingleton" of x . The problem of mixed fusions recurs in Nolan's account as a problem about fusions of \emptyset with things that are not fusions of Esingletons: such things are denied singletons. To spare ordinary objects this fate, Nolan [2004: §4] stipulates that only (certain) 'large' things are eligible as the empty set. Thus again singletons are ensured for all small things.

true iff it is possible iff it is necessary.

The appeal of possibilist structuralism clearly depends on the case for possibilism. If one accepts possibilism anyway – and does not construe them in some mathematical manner –, I think possibilist structuralism is a very attractive position. If, on the other hand, one sees no independent reason at all to accept possibilism, then it is not much of an improvement over traditional Platonism. But even then I believe it is an improvement. For the problems raised by possibilism seem to me less puzzling than those raised by mathematical Platonism. For instance, on Lewis’s [1986] conception, the nature of possible worlds is pretty clear – they are spatiotemporal entities just like our universe –, and no primitive ideology is required to talk about possibilism.¹³ The modal force of statements about possibilism is again explained by the S5 nature of unrestricted modality. As for the epistemological worry, I don’t think knowledge about possibilism raises any further problems than knowledge of modality raises on any conception. Anyway, what is needed for mathematics aren’t any substantial insights into individual essences or the like. All we need to know is that there are sufficiently many possibilism, which, as I’ve argued, is quite plausible.

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¹³ Some conceptions of possible worlds certainly do raise ideological worries. For instance, Rosen’s [1990] fictionalism takes *truth in a fiction* as primitive, and on many ‘abstract’ conceptions it remains mysterious what it means that something *is the case at a given world* (cf. [Lewis 1986: ch.3]).

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