Generalising Kripke Semantics for Quantified Modal Logics

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We turn now to what is arguably one of the least well behaved modal languages ever proposed: first-order modal logic.
[Blackburn and van Benthem 2007]

Contents

1 Introduction . . . 1
2 Counterpart models . . . 3
3 Substitution . . . 13
4 Logics . . . 31
5 Logics with explicit substitution . . . 44
6 Correspondence. . . . 61
7 Canonical models . . . 71
8 Completeness . . . 85
9 Locally classical and two-dimensional logics . . . 87
10 Individualistic semantics . . . 87

1 Introduction

Modal logic has outgrown its philosophical origins. What used to be the logic of possibility and necessity has become topic-neutral, with applications ranging from the validation of computer programs to the study of mathematical proofs. Along the way, *modal predicate logic* has lost its role as the centre of investigation, to the point that it is hardly mentioned in standard textbooks such as [Blackburn et al. 2001]. One reason for this development is that on the standard semantics going back to Kripke, modal predicate logic has turned out to be ill-behaved in several respects. Most notably, applications of the canonical model technique for proving completeness only work for highly restricted classes of models and logical systems, and many interesting systems – e.g., all systems in between quantified S4.3 and S5 ([Ghilardi 1991]) – have been shown to be incomplete, even if their propositional fragment is complete.

Another reason for the decline of modal predicate logic is that propositional modal logic itself has emerged as a fragment of first-order predicate logic, with the domain of "worlds" playing the role of "individuals". As emphasized in [Blackburn et al. 2001], the distinctive character of modal logic is not its subject matter, but its *perspective*. Statements of modal logic describe relational structures from the inside perspective of a particular node. Kripke-style modal predicate logic therefore looks like a somewhat cumbersome hybrid, combining an internal perspective on one class of objects (the domain of the modal operators) with an external perspective on a possibly different class of objects (the domain of quantification).

Nevertheless, this hybrid perspective is useful and natural for many applications – not least the traditional philosophical topics of epistemic, deontic and temporal logic (and the logic of "metaphysical necessity", if there is such a thing). For instance, when reasoning about time, it is natural to take a perspective that is internal to the structure of times, so that what is true at one point may be false at another, but external to the structure of sticks and stones and people existing at the various times.

The main complexities that arise for this approach concern *de re* statements like $\exists x \Diamond Fx$. If we assume that each world or time has an associated domain of individuals that exist at the relevant world or time, we can read $\exists x \Diamond Fx$ as true at w iff w's domain of individuals has a member that satisfies Fx at some point w' accessible from w. Accordingly, if some individual at w does not exist at w', then $\exists x \Diamond \neg \exists y (x = y)$ is true (at w), despite the fact that $\neg \exists y (x = y)$ is not satisfiable in standard predicate logic. This is why modal predicate logics often use free logic as their predicate logic base (see e.g. [Garson 1984]).

A more serious problem for the standard interpretation of modal formulas is that it makes identity non-contingent. For example, if the domain of w contains two individuals x and y, then there can be no other point from the perspective of which these two individuals are identical: $\forall x \forall y (x \neq y \supset \Box x \neq y)$ is valid on the standard semantics. But arguably it is possible to believe of two individuals that they are one and the same, and then we may not want $\forall x \forall y (x \neq y \supset \Box x \neq y)$ to be a theorem of epistemic logic. Moreover, $\forall x \forall y (x \neq y \supset \Box x \neq y)$ is not provable by the standard axioms and rules of first-order logic combined with those of the basic modal logic K. Standard Kripke semantics therefore cannot serve as model theory for one of the simplest modal predicate logics.

David Lewis [1968] once proposed an alternative to Kripke semantics that (among other things) allows for contingent identity. On this account, the *de re* formula $\exists x \Diamond Fx$ is true at w if the domain of w contains an object whose *counterpart* at some accessible point w' satisfies Fx. If two objects at w have the same counterpart at w', $\exists x \exists y (x \neq y \wedge \Diamond (x = y))$ becomes true.

Unfortunately, Lewis combined his proposal with two further ideas, which have prevented its wider adoption in logic and philosophy. First, he held that ordinary individuals never exist at more than one possible world, so that an individual's counterparts at other worlds are never identical to the original individual. The analogous temporal position, defended in [Sider 2001], holds that ordinary individuals are temporarily unextended stages that have other stages as temporal counterparts. This is not how we intuitively think of sticks or stones or persons.

Second, Lewis effectively swapped the traditional, hybrid perspective of Kripke semantics for a thoroughly internal perspective, where statements are evaluated not relative to worlds, but relative to individuals at worlds (see esp. [Lewis 1986: 230-235]). This has curious consequences for the logic determined by his semantics. For instance, the 'necessity of existence', $\Box\exists y(x = y)$, comes out valid, while basic distribution principles such as $\Box(A \land B) \supset \Box A$ become invalid (see [Hazen 1979], [Woollaston 1994]). These consequences can be explained, and the internalist approach has been developed into a powerful model-theoretic framework in the hands of Silvio Ghilardi, Giancarlo Meloni and Giovanna Corsi (see [Ghilardi and Meloni 1988], [Ghilardi and Meloni 1991], [Ghilardi 2001] [Corsi 2002], [Braüner and Ghilardi 2007: 591–616]). Nevertheless, it goes against the traditional conception of modal predicate logic.

In this essay, I will investigate a model theory that combines the familiar picture of Kripke semantics with the idea that individuals are tracked across worlds by a counterpart relation. Since the domain of individuals at different worlds need not be distinct, standard Kripke semantics emerges as the special case where counterparthood is identity. As we will see, lifting this constraint on counterprathood results in a simple and intuitive framework that overcomes several shortcomings of standard Kripke semantics.

My proposal is indebted to [Kutz 2000], which develops a similar framework. ([Kracht and Kutz 2002] summarizes of the main results in English.) Unfortunately, the logics proposed by Kutz are incomplete and his completeness proof invalid. Here these problems will be repaired. Unlike Kutz, I will also offer a model theory for negative free logics without outer domains, and for languages with individual constants and object-language substitution operators. (To incorporate individual constants, Kracht and Kutz [2005] switch from counterpart semantics to what Schurz [2011] calls *worldline semantics*, where quantifiers range over functions from worlds to individuals; see also [Kracht and Kutz 2007].)

2 Counterpart models

Like Kripke semantics, counterpart semantics comes in several flavours. A well-known choice in Kripke semantics is whether the domain of individuals that exist at a world is constant or variable. In counterpart semantics, strict identity across worlds is less important; the more relevant choice is whether every individual that exists at a world should have a counterpart at every other world.

If one allows for individuals to lack counterparts at some worlds, the next question is what can be said about things that don't exist. The alternatives are well-known from free logic. One option is that if x doesn't exist at w, then every atomic predication Fx is false at w. This is known as *negative*, or *single-domain* semantics. Alternatively, one may hold that non-existence is no bar to satisfying predicates, so Fx may be true at some worlds where x doesn't exist and false at others. The extension of F at a world must therefore be specified not only for things that exist at that world, but also for things that don't exist. This is known as *positive*, or *dual-domain* semantics. Both approaches are attractive for certain applications, so I will explore them in tandem.

In positive models, terms are never genuinely empty. Worlds are associated with an *inner domain* of individuals existing at that world, and an *outer domain* of individuals which, although they need not exist, may still fall in the extension of atomic predicates. Every individual at any world has at least one counterpart at every accessible world, if only in the outer domain. In negative models, we want to do without the somewhat ghostly outer domains. Here terms can go genuinely empty. However, in a modal setting, it is not enough to require that whenever x is empty, then atomic predications involving x are false. We must also require that modal predications of the form $\Diamond Fx$ are false for empty x. If $\Diamond Fx$ could be true and $\Diamond Fy$ false although x and y are both empty, we would have to introduce outer domains after all, so that the outer referent of x has an F-counterpart while that of y does not.

As a consequence, the logic of single-domain counterpart models validates some principles that are not derivable from the standard axioms and rules of negative free logic combined with those of the basic modal logic K, notably

(NA) $\neg Ex \supset \Box\neg Ex$,

(TE) $x = y \supset \Box(Ex \supset Ey)$.

Here Ex abbreviates $\exists y(x = y)$. (NA) reflects the fact that non-existent objects don't have any counterparts. (TE) says that if x is identical to y, and x has a counterpart at some accessible world, then y also has a counterpart at that world. If we had outer domains, an individual could have some existing and some non-existing counterparts at a world, which would render (TE) false.

We can offer counterpart models for negative modal predicate logics without (NA) and (TE). These are dual-domain models in which the extension of all predicates, including identity, is restricted to the inner domain. (NA) would then require that individuals which only figure in the outer domain of a world never have counterparts in the inner domain of another world. (TE) would require that if an individual in the inner domain of a world has a counterpart in the inner domain of another world, then all its counterparts at that world are in the inner domain. The two requirements are obviously independent and non-trivial. Hence the axioms (NA) and (TE) are independent of one another and of the standard principles of basic negative free logic combined with K.

Another question which arises specifically for counterpart-theoretic accounts is whether we want to allow for what Allen Hazen calls "internal relations" (see [Hazen 1979: 328–330], [Lewis 1986: 232f.]). Suppose Queen Elizabeth II is necessarily the daughter of George VI. Does that mean that at every world, every counterpart of Elizabeth is the daughter of every counterpart of George? Arguably not, if we want to allow individuals to have multiple counterparts at some worlds. Consider a possible world that embeds two copies of the actual world, a "left" copy and a "right" copy. Plausibly, both Elizabeth II and George VI have two counterparts in this world, but each of the Elizabeth counterparts is only daughter to one of the George counterparts.

To model this sort of situations, we need to allow for different ways of locating the individuals from one world at another world. Formally, we will have multiple counterpart relations. One relation will link Elizabeth and George to their counterparts in the left copy, another to their counterparts in the right copy. $\Box Gab$ will be true iff, *on every counterpart relation*, all counterparts of a G-related to all counterparts of b.[1]

Here is the official definition. As usual, a *model* combines an abstract *structure* with an *interpretation* of our language on that structure. The relevant structures are defined as follows.

> Definition 2.1 (Counterpart structure)
> A *counterpart structure* is a quintuple $\mathcal{S} = \langle W, R, U, D, K \rangle$, consisting of
> 1. a non-empty set W (of "points" or "worlds"),
> 2. a binary ("accessibility") relation R on W,
> 3. a ("outer domain") function U that assigns to each point $w \in W$ a set U_w,
> 4. a ("inner domain") function D that assigns to each point $w \in W$ a set $D_w \subseteq U_w$, and
> 5. a ("counterpart-inducing") function K that assigns to each pair of points $\langle w, w' \rangle \in R$ a non-empty set $K_{w,w'}$ of binary ("counterpart") relations $C \subseteq U_w \times U_{w'}$
>
> such that *either* (i) $D = U$, *or* (ii) all counterpart relations are "total" in the sense that if $C \in K_{w,w'}$, then for each $d \in U_w$ there is a $d' \in U_{w'}$ with dCd'. In case (i), $\mathcal{S}$ is a *single-domain* structure, in case (ii) it is a *total* structure.

1 [Hazen 1979] introduces models with multiple counterpart relations, but stipulates that each relation is actually an injective function, in order to validate the necessity of identity and get a traditional logic of 'actually'. Multiple counterpart relations are also used in [Kutz 2000] and [Kracht and Kutz 2002]. It turns out that the introduction of multiple counterpart relations makes little difference to the base logic. In particular, the logic of all positive or negative counterpart models is exactly the same either way. However, multiple counterpart relations will help in the construction of canonical models for stronger logics in section 7, where we will run into a form of Hazen's "problem of internal relations".

If all counterpart relations are total and also $D = U$, then the structure is both single-domain and total.

At first, the "counterpart-inducing function" with its associated many counterpart relations may look unfamiliar. Think of this as constructed from a Lewisian counterpart relation in two steps. First, we want to drop Lewis's requirement of disjoint domains, so that an individual can occur in the domain of many or all worlds. It is then not enough to just specify which individuals are counterparts of other individuals. For example, we want to allow for d at w to have d' as its only counterpart at w', and d'' at w', although d' also exists at w''. Now is d' a counterpart of d? In effect, counterparthood turns into a four-place relation between one individual at one world and another (or the same) individual at another world. It proves convenient to represent this by associating each pair of worlds with a "local" counterpart relation between the individuals in the associated domains. Secondly, these local counterpart relation give way to sets of relations in order to allow for internal relations.

The following terminology might help to make all this look more familiar. I will say that in a given model, *d' at w' is a counterpart of d at w* iff there is a $C \in K_{w,w'}$ such that dCd'. Similarly, a pair of individuals *$\langle d'_1, d'_2 \rangle$ at w' is a counterpart of $\langle d_1, d_2 \rangle$ at w* iff there is a $C \in K_{w,w'}$ such that $d_1 C d'_1$ and $d_2 C d'_2$. And so on for larger sequences. (Note that an identity pair $\langle d, d \rangle$ at w has $\langle d'_1, d'_2 \rangle$ at w' as counterpart iff there is a $C \in K_{w,w'}$ such that d is C-related to both d'_1 and d'_2.) As we will see, the interpretation of modal formulas can be spelled out in terms of this "counterpart relation" between sequences rather than the function $K_{w,w'}$ on which it is officially based.

That counterparthood should be extended to sequences is suggested in [Lewis 1983] and [Lewis 1986], in response to the problem of internal relations. In the present framework, counterparthood between sequences is a derivative notion. This has the advantage that it immediately rules out some possibilities that would cause trouble. For example, it can never happen that a pair $\langle d_1, d_2 \rangle$ at w has $\langle d'_1, d'_2 \rangle$ at w' as counterpart although by itself, d_1 at w does not have d'_1 at w' as counterpart. Similarly, it can never happen that $\langle d_1, d_2 \rangle$ at w has $\langle d'_1, d'_2 \rangle$ at w' as counterpart while $\langle d_2, d_1 \rangle$ at w does not have $\langle d'_2, d'_1 \rangle$ at w' as counterpart. We also don't have to worry about "gappy" sequences that arise when some things fail to have counterparts. And we automatically get a sensible answer to the question which sequences, in general, matter for the evaluation of a modal formula $\Box A$ at a world – the sequence of individuals denoted by terms in $\Box A$, in order of appearance?, or in alphabetical order of the corresponding terms?, including repetitions if a term occurs more than once?, or should we also consider sequences involving individuals not mentioned in $\Box A$?

Next, we define interpretations of the language of quantified modal logic on counterpart structures. To this end, we first have to say what that language is.

Definition 2.2 (Languages of QML)
A set of formulas $\mathcal{L}$ is a *standard language of quantified modal logic* if there are distinct sets of symbols $\mathit{Var}(\mathcal{L})$ (the *variables* of $\mathcal{L}$), $\mathit{Pred}(\mathcal{L})$ (the *predicates* of $\mathcal{L}$), $\{\neg, \supset, \forall, =, \Box\}$ such that $\mathit{Var}(\mathcal{L})$ is countably infinite and $\mathcal{L}$ is generated by the rule

$$Px_1 \ldots x_n \mid x{=}y \mid \neg A \mid (A \supset B) \mid \forall x A \mid \Box A,$$

where $P \in \mathit{Pred}(\mathcal{L})$, $x, y, x_1, \ldots \in \mathit{Var}(\mathcal{L})$ and every predicate P is limited to a fixed number $n \geq 0$ of variables, called the *arity* of P.

Some notational conventions: I will often use '$\mathcal{L}$' for a fixed but arbitrary language, 'x', 'y', 'z', 'v' (sometimes with indices or dashes) for members of $\mathit{Var}(\mathcal{L})$, and '$F$', '$G$', '$P$' for members of $\mathit{Pred}(\mathcal{L})$ with arity 1, 2 and n, respectively. Formulas involving '$\wedge$', '$\vee$', '$\leftrightarrow$', '$\exists$' and '$\Diamond$' are defined by the usual metalinguistic abbreviations. The order of precedence among connectives is $\neg, \wedge, \vee, \supset$; association is to the right. So $A \wedge B \supset C \supset D$ is $(A \wedge B) \supset (C \supset D)$. For any variable x, 'Ex' abbreviates '$\exists y(y{=}x)$', where y is the alphabetically first variable other than x. '$A_1 \wedge \ldots \wedge A_n$' stands for '$A_1$' if $n = 1$, or for '$(A_1 \wedge \ldots \wedge A_{n-1}) \wedge A_n$' if $n > 1$, or for an arbitrary tautology $\top$ (say, '$x{=}x \supset x{=}x$') if $n = 0$. For any expression or set of expressions A, $\mathit{Var}(A)$ is the set of variables in (members of) A, and $\mathit{Varf}(A)$ is the set of variables with free occurrences in (members of) A.

Definition 2.3 (Interpretation)
Let $\mathcal{S} = \langle W, R, U, D, C \rangle$ be a counterpart structure and $\mathcal{L}$ a language of quantified modal logic. An *interpretation function V for $\mathcal{L}$ on $\mathcal{S}$* is a function that assigns to each world $w \in W$ a function V_w such that
(i) for every predicate P of $\mathcal{L}$, $V_w(P) \subseteq U_w^n$,
(ii) $V_w(=) = \{\langle d, d \rangle : d \in U_w\}$, and
(iii) for every variable x of $\mathcal{L}$, $V_w(x)$ is either undefined or in U_w.

If $V_w(x)$ is undefined for some w and x, then V is called *partial*, otherwise it is *total*.

For zero-ary predicates P, clause (i) says that $V_w(P) \subseteq U_w^0$. For any U_w, there is exactly one "zero-tuple" in U_w^0, which we may identify with the empty set. So U_w^0 has exactly two subsets, the empty set $\emptyset = 0$ and the unit set of the empty set $\{\emptyset\} = 1$. It is convenient to think of these simply as truth-values.

Definition 2.4 (Counterpart model)
A *counterpart model* $\mathcal{M}$ for a language $\mathcal{L}$ consists of a counterpart structure $\mathcal{S}$ together with an interpretation function V for $\mathcal{L}$ on $\mathcal{S}$ such that either $\mathcal{S}$ is single-domain or both $\mathcal{S}$ and V are total. In the first case, $\mathcal{M}$ is a *negative* model; in the second case, it is a *positive* model.

Thus a counterpart model is effectively a collection of free first-order models, with relations R and C that link models and their domains. Since modal operators shift the point of evaluation from one world to another along the accessibility relation R, it never matters what counterparts an individual from one world has at another world unless that world is accessible. This is why I officially stipulated in definition 2.1 that counterparthood is only defined between accessible worlds. As a consequence, one could actually drop the accessibility relation R from structures $\langle W, R, U, D, K \rangle$, since R can be recovered from K: $\langle w, w' \rangle \in R$ iff $\langle w, w' \rangle \in Dom(K)$.

Note that in a negative model, $D_w = U_w$ can be empty. In positive models, D_w may be empty, but U_w must have at least one member, since $V_w(x) \in U_w$.

Variables are non-rigid in the sense that their interpretation is world-relative. However, we will see at the end of this section that the truth-value of a formula A at a world w never depends on what V assigns to variables at worlds $w' \neq w$. For instance, when we evaluate $\Diamond Fx$ at w, we do not check whether Fx is true at some accessible world w', i.e. whether $V_{w'}(x) \in V_{w'}(F)$. Rather, we check whether some individuals at w' that are counterpart-related to $V_w(x)$ are in $V_{w'}(F)$. $V_{w'}(x)$ only enters the picture when we evaluate formulas relative to w'. If we had a designated centre world w_{c} in each model, we could drop the world-relativity of V for individual variables.

Now let's specify how formulas of $\mathcal{L}$ are evaluated at worlds. For the semantics of quantifiers, we need the concept of an x-variant of V.

Definition 2.5 (Variant)
Let V and V' be interpretations on a structure $\mathcal{S}$. V' is an *x-variant of V on w* if V' differs from V at most in the value assigned to x at w. V' is an *existential* x-variant of V on w if in addition, $V'(x) \in D_w$.

$\forall xA$ will then be true at a world w under V iff A is true at w under all existential x-variants V' of V on w. This rule allows us to dispose with assignment functions and to use free variables as individual constants, which makes the semantics slightly simpler. You may have noticed that individual constants are not explicitly mentioned in definition 2.2.

However, unlike e.g. in [Kripke 1963] and [Lewis 1968], the lack of individual constants plays no important role in the present account. Whenever you want to use an individual constant, simply use a variable that never gets bound. If you want, you may also add a clause to the syntax to the effect that a certain class of variables cannot be bound, and call these variables 'individual constants'. Nothing hangs on this way of handling quantifiers. If you prefer a more traditional treatment with assignment functions and a clear separation between constants and variables, it is trivial to translate between the two approaches (see [Bostock 1997: 81–90]).

At worlds considered as counterfactual, variables denote counterparts of the things they originally denoted. So we need an operation that shifts the value of terms to the counterparts of their original value.

> Definition 2.6 (Image)
> Let V and V' be interpretations on a structure $\mathcal{S}$. V' is a *w'-image of V at w* (for short, $V_w \triangleright V'_{w'}$) iff
> (i) for every world w in $\mathcal{S}$ and predicate P, $V_w(P) = V'_w(P)$, and
> (ii) there is a $C \in K_{w,w'}$ such that for every variable x, if $V_w(x)$ is C-related to some $d \in U_{w'}$, then $V'_{w'}(x)$ is some such d, otherwise $V'_{w'}(x)$ is undefined.
> If (i) holds, I will also say that V and V' *agree on all predicates.*

$V'_{w'}(x)$ can only be undefined in negative models. In positive models, this cannot happen because $V_w(x)$ is always defined and counterpart relations are total.

> Definition 2.7 (Truth)
> The relation $w, V \Vdash_{\mathcal{S}} A$ ("A is *true at w in $\mathcal{S}$ under V*") between a world w in a structure $\mathcal{S}$, an interpretation function V on $\mathcal{S}$, and a sentence A is defined as follows.
>
> $w, V \Vdash_{\mathcal{S}} Px_1 \ldots x_n$ iff $\langle V_w(x_1), \ldots, V_w(x_n)\rangle \in V_w(P)$.
> $w, V \Vdash_{\mathcal{S}} \neg A$ iff $w, V \nVdash_{\mathcal{S}} A$.
> $w, V \Vdash_{\mathcal{S}} A \supset B$ iff $w, V \nVdash_{\mathcal{S}} A$ or $w, V \Vdash_{\mathcal{S}} B$.
> $w, V \Vdash_{\mathcal{S}} \forall x A$ iff $w, V' \Vdash_{\mathcal{S}} A$ for all existential x-variants V' of V on w.
> $w, V \Vdash_{\mathcal{S}} \Box A$ iff $w', V' \Vdash_{\mathcal{S}} A$ for all w', V' such that wRw' and $V_w \triangleright V'_{w'}$.
>
> I will drop the subscript $\mathcal{S}$ when the structure is clear from context.

As usual, truth at all worlds in a structure (or a class of structures) under all interpretations is called *validity*.

Definition 2.8 (Validity)
A set of $\mathcal{L}$-formulas Γ is *positively valid* in a set Σ of total counterpart structures if $w, V \Vdash_{\mathcal{S}} A$ for all $A \in \Gamma$, $\mathcal{S} = \langle W, R, U, D, L \rangle \in \Sigma$, $w \in W$ and total interpretations V on $\mathcal{S}$.

Γ is *negatively valid* in a set Σ of single-domain counterpart structures if $w, V \Vdash_{\mathcal{S}} A$ for all $A \in \Gamma$, $\mathcal{S} = \langle W, R, U, D, L \rangle \in \Sigma$, $w \in W$ and interpretations V on $\mathcal{S}$.

Now we can prove that the value that V assigns to variables at other worlds never matters when evaluating formulas at a given world. This follows from the following lemma.

Lemma 2.9 (Coincidence lemma)
Let A be a sentence in a language $\mathcal{L}$ of quantified modal logic, w a world in a structure $\mathcal{S}$, and V, V' interpretations for $\mathcal{L}$ on $\mathcal{S}$ such that V and V' agree on all predicates, and $V_w(x) = V'_w(x)$ for every variable x that is free in A. (In this case, I will say that V and V' *agree at w on the variables in A*.) Then

$$w, V \Vdash_{\mathcal{S}} A \text{ iff } w, V' \Vdash_{\mathcal{S}} A.$$

Proof by induction on A.

1. For atomic formulas, the claim is guaranteed directly by definition 2.7.
2. A is $\neg B$. $w, V \Vdash \neg B$ iff $w, V \nVdash B$ by definition 2.7, iff $w, V' \nVdash B$ by induction hypothesis, iff $w, V' \Vdash \neg B$ by definition 2.7.
3. A is $B \supset C$. $w, V \Vdash B \supset C$ iff $w, V \nVdash B$ or $w, V \Vdash C$ by definition 2.7, iff $w, V' \nVdash B$ or $w, V' \Vdash C$ by induction hypothesis, iff $w, V' \Vdash B \supset C$ by definition 2.7.
4. A is $\forall x B$. By definition 2.7, $w, V \Vdash \forall x B$ iff $w, V^* \Vdash B$ for all existential x-variants V^* of V on w. Each such x-variant V^* agrees at w with the x-variant V'^* of V' on w such that $V'^*(x) = V^*(x)$ on all variables in B. Conversely, each existential x-variant V'^* of V' on w agrees at w with the x-variant V^* of V on w with $V^*(x) = V'^*(x)$ on all variables in B. So by induction hypothesis, $w, V^* \Vdash B$ for all existential x-variants V^* of V on w iff $w, V'^* \Vdash B$ for all existential x-variants V'^* of V' on w, iff $w, V' \Vdash \forall x B$ by definition 2.7.

5. A is $\Box B$. By definition 2.7, $w, V \Vdash \Box B$ iff $w', V^* \Vdash B$ for all w', V^* such that wRw' and $V_w \triangleright V^*_{w'}$, where $V_w \triangleright V^*_{w'}$ means that there is a $C \in K_{w,w'}$ such that for every variable x, either $V_w(x) C V^*_{w'}(x)$ or $V_w(x)$ has no C-counterpart at w' and $V^*_{w'}(x)$ is undefined. Since $V_w(x) = V'_w(x)$ for all variables x, each w'-image of V at w agrees with some w'-image of V' on all variables in B and vice versa. So by induction hypothesis, $w', V^* \Vdash B$ for all w', V^* such that wRw' and $V_w \triangleright V^*_{w'}$ iff $w', V'^* \Vdash B$ for all w', V'^* such that wRw' and $V_w \triangleright V'^*_{w'}$, iff $w, V' \Vdash \Box B$ by definition 2.7. ■

Corollary 2.10 (Locality lemma)
If two interpretations V and V' on a structure $\mathcal{S}$ agree on all predicates and if for all variables x, $V_w(x) = V'_w(x)$, then for any formula A, $w, V \Vdash_{\mathcal{S}} A$ iff $w, V' \Vdash_{\mathcal{S}} A$.

Proof Immediate from lemma 2.9. ■

Negative models can in a sense be "simulated" by positive models: starting with any negative model, we can create a positive model by adding a "null individual" o to the outer domain U_w of every world w; o never satisfies any atomic predicates and serves as referent of previously empty terms.

Definition 2.11 (Positive transpose)
The *positive transpose* $\mathcal{S}^+$ of a counterpart structure $\mathcal{S} = \langle W, R, U, D, K \rangle$ is the structure $\langle W, R, U^+, D, K^+ \rangle$ with U^+ and K^+ constructed as follows. Let o be an arbitrary individual (say, the smallest ordinal) not in $\bigcup_w D_w$. For all $w \in W$, $U^+_w = U_w \cup \{o\}$. For all $\langle w, w' \rangle \in R$, $K^+_{w,w'}$ is the set of relations $C^+ \subseteq U^+_w \times U^+_{w'}$ such that for some $C \in K_{w,w'}$, $C^+ = C \cup \{\langle d, o \rangle : d \in U^+_w$ and there is no $d' \in U_{w'}$ with dCd' $\}$.

If V is a (partial or total) interpretation on a structure $\mathcal{S}$, then the *positive transpose* V^+ of V is the interpretation on $\mathcal{S}^+$ that coincides with V except that whenever $V_w(x)$ is undefined for some w and x, then $V^+_w(x) = o$.

Lemma 2.12 (Truth-preservation under transposes)
If $\mathcal{M}$ is a counterpart model consisting of a structure $\mathcal{S} = \langle W, R, U, D, K \rangle$ and an interpretation V of $\mathcal{L}$ on $\mathcal{S}$, and if $\mathcal{S}^+ = \langle W, R, U^+, D, K^+ \rangle$ and V^+ are the positive transposes of $\mathcal{S}$ and V respectively, then for any world $w \in W$ and formula A of $\mathcal{L}$,

$$w, V \Vdash_{\mathcal{S}} A \text{ iff } w, V^+ \Vdash_{\mathcal{S}^+} A.$$

PROOF by induction on A.

1. A is $Px_1 \dots x_n$. By definition 2.7, $w, V \Vdash_{\mathcal{S}} Px_1 \dots x_n$ iff $\langle V_w(x_1), \dots, V_w(x_n) \rangle \in V_w(P)$. If all $V_w(x_i)$ are defined, then $V^+_w(x_i) = V_w(x_i)$ and $V^+_w(P) = V_w(P)$ by definition 2.11, and so $\langle V_w(x_1), \dots, V_w(x_n) \rangle \in V_w(P)$ iff $\langle V^+_w(x_1), \dots, V^+_w(x_n) \rangle \in V^+_w(P)$, i.e. iff $w, V^+ \Vdash_{\mathcal{S}^+} Px_1 \dots x_n$. If some $V_w(x_i)$ is undefined, then $\langle V_w(x_1), \dots, V_w(x_n) \rangle$ is undefined and not in $V_w(P)$; so $w, V \nVdash_{\mathcal{S}} Px_1 \dots x_n$. Moreover, in this case $V^+_w(x_i) = o$ and since $V_w(P) = V^+_w(P)$ never contains any tuples involving o, $\langle V^+_w(x_1), \dots, V^+_w(x_n) \rangle \notin V^+_w(P)$. So either way, $w, V \nVdash_{\mathcal{S}} Px_1 \dots x_n$ iff $w, V^+ \Vdash_{\mathcal{S}^+} Px_1 \dots x_n$.

2. A is $\neg B$. $w, V \Vdash_{\mathcal{S}} \neg B$ iff $w, V \nVdash_{\mathcal{S}} B$ by definition 2.7, iff $w, V^+ \nVdash_{\mathcal{S}^+} B$ by induction hypothesis, iff $w, V^+ \Vdash_{\mathcal{S}^+} \neg B$ by definition 2.7.

3. A is $B \supset C$. $w, V \Vdash_{\mathcal{S}} B \supset C$ iff $w, V \nVdash_{\mathcal{S}} B$ or $w, V \Vdash_{\mathcal{S}} C$ by definition 2.7, iff $w, V^+ \nVdash_{\mathcal{S}^+} B$ or $w, V^+ \Vdash_{\mathcal{S}^+} C$ by induction hypothesis, iff $w, V^+ \Vdash_{\mathcal{S}^+} B \supset C$ by definition 2.7.

4. A is $\forall x B$. By definition 2.7, $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, V' \Vdash_{\mathcal{S}} B$ for all existential x-variants V' of V on w. These x-variants V' of V correspond one-one to the x-variants ${V^+}'$ of V^+ with ${V^+}'_w(x) = V'_w(x)$. Moreover, for each such pair $V', {V^+}'$, $\langle \mathcal{S}^+, {V^+}' \rangle$ is the positive transpose of $\langle \mathcal{S}, V' \rangle$. By induction hypothesis, $w, V' \Vdash_{\mathcal{S}} B$ iff $w, {V^+}' \Vdash_{\mathcal{S}^+} B$. And so $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, {V^+}' \Vdash_{\mathcal{S}^+} B$ for all existential x-variants ${V^+}'$ of V^+ on w, iff $w, V^+ \Vdash_{\mathcal{S}^+} \forall x B$ by definition 2.7.

5. A is $\Box B$. Assume $w, V \Vdash_{\mathcal{S}} \Box B$. By definition 2.7, this means that $w', V' \Vdash_{\mathcal{S}} B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. We need to show that $w', {V^+}' \Vdash_{\mathcal{S}^+} B$ for all $w', {V^+}'$ with wRw' and $V^+_w \triangleright {V^+}'_{w'}$. So let $w', {V^+}'$ be such that wRw' and $V^+_w \triangleright {V^+}'_{w'}$. Since V^+ is a total interpretation and $\mathcal{S}^+$ a total structure, this means that for every variable x there is a $C^+ \in K^+_{w,w'}$ with $V^+_w(x) C^+ {V^+}'_{w'}(x)$. Let V' be the interpretation on $\mathcal{S}$ that coincides with ${V^+}'$ except that $V'_w(x)$ is undefined for every world $w \in W$ and variable x for which ${V^+}'_w(x) = o$. Let $C = \{\langle d, d' \rangle \in C : d \neq o \text{ and } d' \neq o\}$. Now assume there are d, d' with $V_w(x) = d$ and dCd'. Then $V^+_w(x) = d$ and dC^+d' and thus ${V^+}'_{w'}(x) \neq o$, for $\langle d, o \rangle \in C^+$ only if there is no d' with $\langle d, d' \rangle \in C$. So $V'_{w'}(x) = {V^+}'_{w'}(x)$, and $V_w(x) C V'_{w'}(x)$. On the other hand, assume there are no d, d' with $V_w(x) = d$ and dCd', either because $V_w(x)$ is undefined or because $V_w(x) = d$ and the only d' with $\langle d, d' \rangle \in C^+$ is o. Either way, then ${V^+}'_{w'}(x) = o$, and so $V'_{w'}(x)$ is undefined. So for all variables x, if there are d, d' with $V_w(x) = d$ and dCd' then $V_w(x) C V'_{w'}(x)$, otherwise $V'_{w'}(x)$ is undefined. Since $C \in K_{w,w'}$ by construction of K^+ (definition 2.11), this means that $V_w \triangleright V'_{w'}$. But ${V^+}'$ is the positive transpose of V'. So we've shown that whenever $V^+_w \triangleright {V^+}'_{w'}$, then there is a V' such that ${V^+}'$ is the positive transpose of V' and $V_w \triangleright V'_{w'}$. We know that $w', V' \Vdash_{\mathcal{S}} B$. So by induction hypothesis, $w', {V^+}' \Vdash_{\mathcal{S}^+} B$. That is, for each $w', {V^+}'$ with wRw' and $V^+_w \triangleright {V^+}'_{w'}$, $w', {V^+}' \Vdash_{\mathcal{S}^+} B$. By definition 2.7, this means that $w, V^+ \Vdash_{\mathcal{S}^+} \Box B$.

   In the other direction, assume $w, V^+ \Vdash_{\mathcal{S}^+} \Box B$. That is, $w', {V^+}' \Vdash_{\mathcal{S}^+} B$ for each $w', {V^+}'$ with wRw' and $V^+_w \triangleright {V^+}'_{w'}$. We have to show that $w', V' \Vdash_{\mathcal{S}} B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. So let w', V' be such that wRw' and $V_w \triangleright V'_{w'}$. The latter means that there is a $C \in K_{w,w'}$ such that for every variable x, either $V_w(x) C V'_{w'}(x)$ or $V_w(x)$ has no C-counterpart at w' and $V'_{w'}(x)$ is undefined. Let V'^+ be the positive transform of V'. Let $C^+ = C \cup \{\langle d, o\rangle : d \in U_w^+$ and there is no $d' \in U_{w'}$ with dCd' $\}$. By definition 2.11, $C^+ \in K^+_{w,w'}$. For any variable x, if $V_w(x)CV'_{w'}(x)$, then both $V_w(x)$ and $V'_{w'}(x)$ are defined and thus $V_w^+(x) = V_w(x)$ and $V'^+_{w'}(x) = V'_{w'}(x)$ by definition 2.11; moreover, then $V_w^+(x)C^+V'^+_{w'}(x)$ since $C \subseteq C^+$. On the other hand, if $V_w(x)$ has no C-counterpart at w', so that $V'_{w'}(x)$ is undefined, then by construction of C^+ and V_w^+, $V_w^+(x)$ (which equals $V_w(x)$ if $V_w(x)$ is defined, else o) has o as C^+-counterpart at w'; and $V'^+_{w'}(x) = o$; so again $V_w^+(x)C^+V'^+_{w'}(x)$. So for every variable x, there is a $C^+ \in K^+_{w,w'}$ with $V_w^+(x)C^+V'^+_{w'}(x)$, and so $V_w^+ \triangleright V'^+_{w'}$. Now we know that $w', V^{+\prime} \Vdash_{\mathcal{S}^+} B$ for all $w', V^{+\prime}$ with wRw' and $V_w^+ \triangleright V^{+\prime}_{w'}$. Hence $w', V'^+ \Vdash_{\mathcal{S}^+} B$. By induction hypothesis, $w', V' \Vdash_{\mathcal{S}} B$. So we've shown that whenever wRw' and $V_w \triangleright V'_{w'}$, then $w', V' \Vdash_{\mathcal{S}} B$. By definition 2.7, this means that $w, V \Vdash_{\mathcal{S}} \Box B$. ∎

3 Substitution

Before we look at logical systems for our models, we need to talk a little bit about substitution.

Modal operators shift the point of evaluation: whether $\Box A$ or $\Diamond A$ is true at w often depends on what is true at other points w'. In counterpart semantics, when the point of evaluation is shifted from w to w', the semantic value of every individual constant and variable shifts to the counterpart of the previous value, following some counterpart relation C. If an individual at w has no C-counterpart at w', the relevant terms become empty. If an individual has multiple C-counterparts, we might think of the corresponding terms as becoming "ambiguous", denoting all the counterparts at the same time. To verify $\Box Fx$, we require that Fx is true at all accessible worlds under all "disambiguations".

An important question now is whether these disambiguations are uniform or mixed: should $\Box Gxx$ be true iff at all accessible worlds (for all counterpart relations C) all x counterparts are G-related to *themselves* (uniform) or to *one another* (mixed)? On the mixed account, $\Box x{=}x$ becomes invalid, as does $\Box(Fx \lor \neg Fx)$, even if x exists at all worlds. Moreover, the semantics becomes more complicated because a mixed disambiguation cannot be represented by a standard interpretation function; so if we say that $\Box A$ is true relative to interpretation V iff A is true at all accessible worlds under all interpretation functions V' suitably related to V, we automatically get uniform disambiguations. Thus I have used uniform disambiguations in the previous section. (See section **??** for the alternative route.)

The present issue might remind you of the old observation that a sentence like 'Brutus killed himself' can be understood either as an application of a *monadic* predicate 'killing himself' to the subject Brutus, or as an application of the binary 'killing' to Brutus and Brutus. Peter Geach once suggested a syntactic mechanism for distinguishing these readings, by introducing an operator $\langle z : x, y\rangle$ that turns a binary expression into a unary expression: while Gxy is satisfied by pairs of individuals as values of x and y, $\langle z : x, y\rangle Gxy$ is satisfied by a single individual for z. The operator $\langle z : x, y\rangle$, which might be read 'z is an x and a y such that' acts as a quantifier that binds both x and y.

A similar trick can be used in our modal context. On the uniform reading, $\Box x = x$ says that all counterparts of x are self-identical at all accessible worlds. To say that at all accessible worlds (and under all counterpart relations), all x-counterparts are identical to all x-counterparts we could instead say $\langle x : y, z\rangle \Box y = z$. The effect of $\langle x : y, z\rangle$ is to introduce two variables y and z that co-refer with x. By using distinct but co-refering variables in a modal context, we can express relations between possibly distinct counterparts; by using the same variable, we make sure that the same counterpart must be assigned to every occurrence.

With $\langle x : y, z\rangle \Box y = z$, we actually end up with *three* co-referring variables: y and z are made to co-refer with x, but we also have x itself. Thus the job can also be done with $\langle x : y\rangle \Box x = y$ – read: 'x is a y such that ...'.

To see the use of this operator, consider the following two sentences, which look at first glance like simple applications of universal instantiation.

(1) $\forall x \Box Gxy \supset \Box Gyy$;

(2) $\forall x \Diamond Gxy \supset \Diamond Gyy$.

Suppose for a moment that we have at most one counterpart relation from any world to another, so that we can ignore the quantification over counterpart relations. The first formula then says that if all things x are such that all x-counterparts are G-related to all y-counterparts, then all y-counterparts are G-related to themselves. That must be true. (2), however, is not valid. If all things x are such that some x-counterpart is G-related to some y-counterpart, it only follows that some y-counterpart is G-related to some y-counterpart; it does not follow that some y-counterpart is G-related to itself.

With the two distinct variables x and y, the formula $\Diamond Gxy$ looks at arbitrary combinations of x-counterparts and y-counterparts, even if $x = y$. By contrast, $\Diamond Gyy$ only looks at single y counterparts and checks whether one of them is G-related to itself. To prevent this accidental "capturing" of y in the consequent of (2), we can use the Geach quantifier:

(2′) $\forall x \Diamond Gxy \supset \langle y : x\rangle \Diamond Gxy$

Having multiple counterpart relations makes no essential difference to these considerations. Since Gyy is a formula of type $A(y)$, $\Diamond Gyy$ is true at w iff Gyy is true at some accessible world under some assignment of a y-counterpart to 'y'. On the other hand, given $x = y$, $\Diamond Gxy$ is true at w iff there is an accessible world at which some counterpart of the pair $\langle x, y\rangle$ $(= \langle x, x\rangle = \langle y, y\rangle)$ satisfies Gxy.

The same issues arise for Leibniz' Law. In the pair

(3) $x = y \supset \Box Gxy \supset \Box Gyy$;

(4) $x = y \supset \Diamond Gxy \supset \Diamond Gyy$,

only the first sentence is valid. In (4), the substituted variable y again gets captured by the other occurrence of y in the scope of the diamond. To avoid this, we should write

(4′) $x = y \supset \Diamond Gxy \supset \langle y : x\rangle \Diamond Gxy$

in place of (4). The Geach quantifier $\langle y : x\rangle$ thus functions as an *object-language substitution operator*.

If we want to use this operator, we have to extend the language of quantified modal logic.

Definition 3.1 (Languages of QML with substitution)
A *language of quantified modal logic with substitution* is like a standard language of quantified modal logic (definition 2.2) except that there is a further logical constant $\langle : \rangle$ with the construction rule that whenever x, y are variables and A is a formula of $\mathcal{L}$, then $\langle y : x\rangle A$ is a formula of $\mathcal{L}$.

As for the semantics: just as $\forall x A$ is true relative to an interpretation V iff A is true relative to all x-variants of V (on the relevant domain), $\langle y : x\rangle A$ is true relative to V iff A is true relative to the x-variant of V that maps x to $V(y)$. In our modal framework:

Definition 3.2 (Semantics for the substitution operator)
$w, V \Vdash \langle y : x\rangle A$ iff $w, V' \Vdash A$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$.

Note that V' need not be an *existential* x-variant of V on w.

The coincidence lemma 2.9 is easily adjusted to languages with substitution, and corollary 2.10 follows as before. I won't go through the whole proof again. Here is the only new step in the induction:

> A is $\langle y : x\rangle B$. $w, V \Vdash \langle y : x\rangle B$ iff $w, V^* \Vdash B$ where V^* is the x-variant of V on w with $V^*_w(x) = V_w(y)$. Let V'^* be the x-variant of V' on w with $V'^*_w(x) = V'_w(y)$. Then V^* and V'^* agree at w on all variables in B, so by induction hypothesis, $w, V^* \Vdash B$ iff $V'^*, w \Vdash B$. And this holds iff $w, V' \Vdash \langle y : x\rangle B$ by the semantics of $\langle y : x\rangle$. ■

Substitution operators turn out to have significant expressive power. As [Kuhn 1980] shows (in effect), if a language has substitution operators, it no longer needs variables or individual constants in its atomic formulas: instead of Fx, we can simply say F, with the convention that the implicit variable is always x (more generally, the first one is x, the second y, etc.); Fy turns into $\langle y : x \rangle F$, Gyz into $\langle y : x \rangle \langle z : y \rangle G$. Similarly, $\forall x Fx$ can be replaced by $\forall F$, and $\forall y Gxy$ by $\forall \langle y : z \rangle \langle x : y \rangle \langle z : x \rangle G$. So we also don't need different quantifiers for different variables any more. In this essay, however, I will not exploit this power of substitution operators – mainly for the sake of familiarity. Our languages with substitution operators will still have ordinary formulas $Px_1 \ldots x_n$ and quantifiers $\forall x, \forall y$, etc.

Substitution operators are close cousins of lambda abstraction and application, as introduced to modal logic in [Stalnaker and Thomason 1968]. Lambda abstraction converts a formula A into a predicate $(\lambda x.A)$, which can then be applied to a singular term y to form a new formula $(\lambda x.A)y$. Semantically, $(\lambda x.A)y$ is true under an interpretation V at a world w iff A is true under the x-variant V' of V on w with $V'_w(y) = V_w(x)$. Thus $(\lambda x.A)y$ is just another way of writing $\langle y : x \rangle A$.

The standard use of lambda abstraction in modal logic is to resolve an ambiguity in modal formulas with non-rigid terms. In English, sentences like 'the pope could have been Italian' have a *de dicto* reading and a *de re* reading. On the *de dicto* reading, the sentence says that it could have been that whoever is pope is Italian; on the *de re* reading, the sentence says of the actual pope that *he* could have been Italian. The ambiguity arises because 'the pope' is a non-rigid designator: relative to every world w, it picks out whoever is pope at w. If α is such a non-rigid designator, then the obvious interpretation of $\Diamond F\alpha$ is the *de dicto* reading: $\Diamond F\alpha$ is true at w iff for some accessible world w', the referent of α at w' falls in the extension of F at w'. To express the *de re* reading, lambda abstraction can be used: $(\lambda x.\Diamond Fx)\alpha$.

In counterpart semantics, the distinction between rigid and non-rigid designators is less straightforward than in standard Kripke semantics. By the rules of the previous section, $\Diamond Fx$ is true at w under interpretation V iff some counterpart of $V_w(x)$ at some accessible w' is in $V_{w'}(F)$. If counterparthood is identity, then x functions as a standard rigid designator. On the other hand, if counterparthood is defined so the pope at w has all and only the popes at other worlds as counterpart, and $V_w(x)$ is the pope at w, then x looks much like 'the pope' in English. This is not a defect. A common argument for counterpart treatments of ordinary modal discourse is precisely that there are multiple legitimate ways of re-identifying an individual at our world at other possible worlds (see e.g. [Lewis 1986: 251–255]).

On the other hand, it would be implausible to explain the ambiguity of 'the pope could have been Italian' as an ambiguity in the counterparthood relation associated with 'the pope'. After all, there is no such ambiguity in 'Jorge Mario Bergoglio could have been an

Italian'. Definite descriptions like 'the pope' are non-rigid in a sense that does not carry over to names like 'Jorge Mario Bergoglio'. If we wanted to add non-rigid singular terms α that function like 'the pope' to our modal language, we should say that $\Diamond F\alpha$ is true at w under V iff there is some accessible w' for which $V_{w'}(\alpha) \in V_{w'}(F)$: on its non-rigid interpretation, the referent of 'the pope' at a world w' does not depend on its referent at the actual world w; the term is in a sense directly evaluated at w'.

By the semantics of the previous section, all designators are in this sense rigid. Thus definite descriptions like 'the pope' are not adequately formalised as individual constants in our modal language. We could add a new category of non-rigid "constants", but I will not do consider such an expansion, partly for the sake of simplicity and partly because I think an adequate treatment of descriptions should treat them as composites of a determiner ('the') and a predicate, rather than as atomic terms.

So we don't need lambda abstraction or substitution operators to distinguish *de dicto* and *de re*, because everything is always *de re*. Our main use of substitution operators is rather to distinguish between "mixed" and "uniform" assignments of counterparts. This comes in handy when we look at substitution principles such as Leibniz's Law.

It is well-known that in first-order logic, careless substitution of variables can cause accidental capturing. For example,

(5) $\quad x = y \supset (\exists y(x \neq y) \supset \exists y(y \neq y))$

is not a valid instance of Leibniz's Law, because the variable y gets captured by the quantifier $\exists y$. There are two common ways to respond. One is to define substitution as a simple replacement of variables, and restrict principles like Leibniz's Law:

(LL$^-$) $\; x = y \supset (A \supset [y/x]A)$ provided x is free in A and y is free for x in A,

where y is *free for x in A* if no occurrence of x in A lies in the scope of a y-quantifier. The other response is to use a more sophisticated definition of substitution on which $\exists y(y \neq y)$ does not count as a proper substitution instance of $\exists y(x \neq y)$.

Informally, $[y/x]A$ should say about y exactly what A says about x. More precisely, proper substitutions should satisfy the following condition, sometimes called the "substitution lemma":

$$w, V \Vdash [y/x]A \text{ iff } w, V^{[y/x]} \Vdash A,$$

where $V^{[y/x]}$ is like V except that it assigns to x (at any world) the value V assigns to y.

This goal can be achieved by applying the substitution to an alphabetic variant of the original formula in which the bound variables have been renamed so that capturing can't happen. Thus before x is replaced by y in $\exists y(x \neq y)$, the variable y is made free for x by renaming all bound occurrences of y by some new variable z. Instead of (5), a legitimate instance of Leibniz's Law then is

(5′) $\quad x = y \supset (\exists y(x \neq y) \supset \exists z(y \neq z))$.

The following definition uses this idea to prevent capturing of variables by quantifiers.[2]

Definition 3.3 (Classical Substitution)
For any variables x, y, z let $[y/x]z$ be y if $z = x$ (i.e., z is the same variable as x), otherwise $[y/x]z$ is z. For formulas A, define $[y/x]A$ as A if $x = y$; otherwise

$$
\begin{aligned}
&[y/x]Px_1 \dots x_n &&= P[y/x]x_1 \dots [y/x]x_n\\
&[y/x]\neg A &&= \neg[y/x]A;\\
&[y/x](A \supset B) &&= [y/x]A \supset [y/x]B;\\
&[y/x]\forall zA &&= \begin{cases} \forall v[y/x][v/z]A & \text{if } z = y \text{ and } x \in \mathit{Varf}(A), \text{ or } z = x \text{ and } y \in \mathit{Varf}(A),\\ \forall[y/x]z[y/x]A & \text{otherwise,} \end{cases}\\
& &&\text{where } v \text{ is the alphabetically first variable not in } \mathit{Var}(A), x, y.\\
&[y/x]\langle y_2 : z\rangle A &&= \begin{cases} \langle [y/x]y_2 : v\rangle[y/x][v/z]A & \text{if } z = y \text{ and } x \in \mathit{Varf}(A) \text{ or } z = x \text{ and } y \in \mathit{Varf}(A),\\ \langle [y/x]y_2 : [y/x]z\rangle[y/x]A & \text{otherwise,} \end{cases}\\
& &&\text{where } v \text{ is the alphabetically first variable not in } \mathit{Var}(A), x, y, y_2.\\
&[y/x]\Box A &&= \Box[y/x]A.
\end{aligned}
$$

In standard predicate logic, the last two clauses are empty because there is neither a box nor a substitution quantifier; I've added them because we'll need them later. The clauses for the substitution quantifier are exactly parallel to those for the universal quantifier, and the underlying motivation is the same. For example, $[y/x]\langle y_2 : y\rangle x \neq y$ is $\langle y_2 : z\rangle y \neq z$ rather than $\langle y_2 : y\rangle y \neq y$.

The clause for the box treats substitution into modal contexts as unproblematic. However, as Lewis [1983] observed, in counterpart semantics modal operators effectively function as unselective binders that capture all variables in their scope. As a consequence, our definition of substitution does not satisfy the substitution lemma. For instance, $w, V^{[y/x]} \Vdash \Diamond Gxy$ does not imply $w, V \Vdash \Diamond Gyy$. Informally, $\Diamond Gxy$ says about the individual x that at some world, one of its counterparts is G-related to some y-counterpart;

2 See [Bell and Machover 1977: 54-67], [Gabbay et al. 2009: 87–103] for alternative definitions with the same goal. An unconventional aspect of the present definition is that $[y/x]A$ can replace bound occurrences of x. For example, $[y/x]\forall xFx$ is $\forall yFy$. This has the advantage of generalising nicely to polyadic substitutions and especially transformations, defined below.

$\Diamond Gyy$ does not say the same thing about y, for it says that at some world, one of y's counterparts is G-related to *itself*.[3]

The situation is then similar to that of naive substitution in first-order logic. We can either restrict principles like Leibniz's Law or adjust the definition of substitution so that it satisfies the substitution lemma.

A suitable restriction for substitution principles is that when y is substituted for x in A, y must be *modally free for* x in A, in the following sense.

> DEFINITION 3.4 (MODAL SEPARATION AND MODAL FREEDOM)
> Two variables x and y are *modally separated in* a formula A if no free occurrences of x and y in A lie in the scope of the same modal operator.
>
> y is *modally free for x in A* if either (i) $x = y$, or (ii) x and y are modally separated in A, or (iii) A has the form $\Box B$ and y is modally free for x in B.

For example, x and y are modally separated in $\Box Fx \supset \Diamond Fy$ and in $\forall x \Box Gxy$. y is modally free for x in $\Box x = y$ and $\Box\Box\neg Gxy$ and $\Box\Diamond\neg\exists xGxy$, but not in $\Box\Diamond\neg Gxy$. Correspondingly,

$$x = y \supset (\Box x = y \supset \Box y = y) \text{ and}$$
$$x = y \supset (\Box\Box\neg Gxy \supset \Box\Box\neg Gyy) \text{ and}$$
$$x = y \supset (\Box\Box\neg\exists xGxy \supset \Box\Box\neg\exists zGyz)$$

are valid, while

$$x = y \supset (\Box\Diamond\neg Gxy \supset \Box\Diamond\neg Gyy)$$

is invalid.

If we don't want to restrict the substitution principles, can we redefine substitution so that it satisfies the substitution lemma? In standard languages of quantified modal logic, this is not easy, as we will see in a moment, after we've spelled out some special conditions under which classical substitution (as defined in definition 3.11) does its job even in counterpart models.

---

3 People sometimes complain that counterpart semantics doesn't validate the necessity of identity

(NI) $\quad x = y \supset (\Box x = x \supset \Box x = y)$,

which is allegedly an instance of Leibniz's Law. But you can't just look at the shape of a formula to see whether it is a legitimate instance of Leibniz's Law. If $\Box$ is synonymous to $\forall y$, then (NI) is a tautological variant of (5) and clearly invalid. Similarly, there can be no argument about whether (NI) is a legitimate instance of Leibniz's Law in counterpart semantics. It definitely isn't.

Definition 3.5 (Interpretation under substitution)
For any interpretation V of a language $\mathcal{L}$ on a structure $\mathcal{S}$ and variables x, y of $\mathcal{L}$, $V^{[y/x]}$ is the interpretation that is like V except that for any world w in $\mathcal{S}$, $V^{[y/x]}_w(x) = V_w(y)$.

Lemma 3.6 (Restricted substitution lemma, preliminary version)
Let A be a sentence in a language $\mathcal{L}$ (with or without substitution), $\mathcal{S}$ a counterpart structure for $\mathcal{L}$, w a world in $\mathcal{S}$, V an interpretation on $\mathcal{S}$. Then

$$w, V^{[y/x]} \Vdash_{\mathcal{S}} A \text{ iff } w, V \Vdash_{\mathcal{S}} [y/x]A, \text{ provided } y \notin \mathit{Var}(A).$$

Proof If $y = x$, then $[y/x]A = A$ and $V^{[y/x]} = V$, so the result is trivial. Assume then that $y \neq x$. The proof is by induction on A.

1. A is $Px_1 \dots x_n$. $w, V^{[y/x]} \Vdash Px_1 \dots x_n$ iff $\langle V^{[y/x]}_w(x_1), \dots, V^{[y/x]}_w(x_n)\rangle \in V^{[y/x]}_w(P)$ by definition 2.7, iff $\langle V_w([y/x]x_1), \dots, V_w([y/x]x_n)\rangle \in V_w(P)$ by definition 3.5, iff $w, V \Vdash P[y/x]x_1 \dots [y/x]x_n$ by definition 2.7, iff $w, V \Vdash [y/x]Px_1 \dots x_n$ by definition 3.3.

2. A is $\neg B$. $w, V^{[y/x]} \Vdash \neg B$ iff $w, V^{[y/x]} \nVdash B$ by definition 2.7, iff $w, V \nVdash [y/x]B$ by induction hypothesis, iff $w, V \Vdash [y/x]\neg B$ by definitions 2.7 and 3.3.

3. A is $B \supset C$. $w, V^{[y/x]} \Vdash B \supset C$ iff $w, V^{[y/x]} \nVdash B$ or $w, V^{[y/x]} \Vdash C$ By definition 2.7, iff $w, V \nVdash [y/x]B$ or $w, V \Vdash [y/x]C$ by induction hypothesis, iff $w, V \Vdash [y/x](B \supset C)$ by definitions 2.7 and 3.3.

4. A is $\forall zB$. Since $y \notin \mathit{Var}(A)$, $[y/x]A = \forall[y/x]z[y/x]B$. Assume first that $z \neq x$. By definition 2.7, $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V^{[y/x]\prime} \Vdash B$ for all existential z-variants $V^{[y/x]\prime}$ of $V^{[y/x]}$ on w. These $V^{[y/x]\prime}$ are precisely the functions $V'^{[y/x]}$ where V' is an existential z-variant of V on w. So, $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V'^{[y/x]} \Vdash B$ for all existential z-variants V' of V on w. By induction hypothesis, $w, V'^{[y/x]} \Vdash B$ iff $w, V' \Vdash [y/x]B$. So $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V' \Vdash \forall z[y/x]B$ by definition 2.7, iff $w, V \Vdash [y/x]\forall zB$ by definition 3.3.

   Alternatively, assume $z = x$. By definition 3.3, $[y/x]\forall zB$ is $\forall y[y/x]B$. Assume $w, V \nVdash \forall y[y/x]B$. By definition 2.7, then $w, V' \nVdash [y/x]B$ for some existential y-variant V' of V on w. By induction hypothesis, then $w, V'^{[y/x]} \nVdash B$. Let V^* be the (existential) x-variant of V on w with $V^*_w(x) = V'^{[y/x]}_w(x) = V'_w(y)$. V^* is a y-variant on w of $V'^{[y/x]}$, and y is not free in B, so by the coincidence lemma 2.9, $w, V^* \nVdash B$. But V^* is also an existential x-variant of $V^{[y/x]}$ on w. So $w, V^{[y/x]} \nVdash \forall xB$ by definition 2.7.

   Conversely, assume $w, V^{[y/x]} \nVdash \forall xB$. By definition 2.7, then $w, V^* \nVdash B$ for some existential x-variant of $V^{[y/x]}$ (and thus V) on w. Let V' be the (existential) y-variant of V on w with $V'_w(y) = V^*_w(x)$. Then $V'^{[y/x]}$ and V^* agree at w on all variables except y; in particular, $V'^{[y/x]}_w(x) = V'_w(y) = V^*_w(x)$. Since y is not free in B, by the coincidence lemma 2.9, $w, V'^{[y/x]} \nVdash B$. By induction hypothesis, $w, V' \nVdash [y/x]B$. And since V' is an existential y-variant of V on w, then $w, V \nVdash \forall y[y/x]B$ by definition 2.7.

5. A is $\langle y_2 : z\rangle B$. Since $y \notin Var(A)$, $[y/x]A = \langle [y/x]y_2 : [y/x]z\rangle [y/x]B$ by definition 3.3. By definition 3.2, $w, V \Vdash \langle [y/x]y_2 : [y/x]z\rangle [y/x]B$ iff $w, V' \Vdash [y/x]B$, where V' is the $[y/x]z$-variant of V on w with $V'_w([y/x]z) = V_w([y/x]y_2)$. By induction hypothesis, $w, V' \Vdash [y/x]B$ iff $w, V'^{[y/x]} \Vdash B$. Let $V^{[y/x]\prime}$ be the z-variant of $V^{[y/x]}$ on w with $V^{[y/x]\prime}_w(z) = V^{[y/x]}_w(y_2)$. Then $V^{[y/x]\prime}$ and $V'^{[y/x]}$ agree at w about all variables v in A: for $v = z$, $V^{[y/x]\prime}_w(z) = V_w([y/x]y_2) = V'_w([y/x]z) = V'^{[y/x]}_w(z)$; for $v = x \neq z$, $V^{[y/x]\prime}_w(x) = V^{[y/x]}_w(x) = V_w(y) = V'_w(y)$ (because $y \neq z$ and hence $[y/x]z \neq y$) $= V'_w([y/x]x) = V'^{[y/x]}_w(x)$; and for $v \notin \{x, y, z\}$, $V^{[y/x]\prime}_w(v) = V^{[y/x]}_w(v) = V_w(v) = V'_w(v) = V'^{[y/x]}_w(v)$. So by the coincidence lemma 2.9, $w, V'^{[y/x]} \Vdash B$ iff $w, V^{[y/x]\prime} \Vdash B$. And by definition 3.2, $w, V^{[y/x]\prime} \Vdash B$ iff $w, V^{[y/x]} \Vdash \langle y_2 : z\rangle B$.

6. A is $\Box B$. By definition 2.7, $w, V^{[y/x]} \Vdash \Box B$ iff $w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ with wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. On the other hand, $w, V \Vdash [y/x]\Box B$ iff $w, V \Vdash \Box[y/x]B$ (by definition 3.3), iff $w', V' \Vdash [y/x]B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. By induction hypothesis, $w', V'^{[y/x]} \Vdash B$ iff $w', V' \Vdash [y/x]B$. So we have to show that

(1) $\quad w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ such that wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$

iff

(2) $\quad w', V'^{[y/x]} \Vdash B$ for all w', V' such that wRw' and $V_w \triangleright V'_{w'}$.

(1) implies (2) because every interpretation $V'^{[y/x]}$ with $V_w \triangleright V'_{w'}$ is also an interpretation $V^{[y/x]\prime}$ with $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$.

Assume y is not free in $\Box B$, and that (2) holds. In order to derive (1), consider any w'-image $V^{[y/x]\prime}$ of $V^{[y/x]}$ at w. I.e., there is a counterpart relation $C \in K_{w,w'}$ such that for all z, $V^{[y/x]\prime}_{w'}(z)$ is a C-counterpart of $V^{[y/x]}_w(z)$. Let V^* be like $V^{[y/x]\prime}$ except that $V^*_{w'}(y) = V^{[y/x]\prime}_{w'}(x)$. Let V' be like V^* except that $V'_{w'}(x)$ is some C-counterpart of $V_w(x)$, or undefined if there is none. Then $V_w \triangleright V'_{w'}$. (In particular, $V'_{w'}(y) = V^*_{w'}(y) = V^{[y/x]\prime}(x)$ is a C-counterpart of $V^{[y/x]}_w(x) = V_w(y)$, or undefined if there is none.) So by (2), $w', V'^{[y/x]} \Vdash B$. But $V'^{[y/x]} = V^*$ (since $V^*_{w'}(x) = V^*_{w'}(y)$). So $w', V^* \Vdash B$. And since y is not free in B and V^* is a y-variant of $V^{[y/x]\prime}$ on w', by the coincidence lemma 2.9, $w', V^* \Vdash B$ iff $w', V^{[y/x]\prime} \Vdash B$. ■

A stronger version of this will be proved as lemma 3.9 below. We only need this version to verify that renaming bound variables (a.k.a. α-conversion) does not affect truth-values.

Definition 3.7 (Alphabetic variant)
A formula A' of a language of quantified modal logic (with or without substitution) is an *alphabetic variant of* a formula A if one of the following conditions is satisfied.

1. $A = A'$.
2. $A = \neg B$, $A' = \neg B'$, and B' is an alphabetic variant of B.
3. $A = B \supset C$, $A' = B' \supset C'$, and B', C' are alphabetic variants of B, C, respectively.
4. $A = \forall x B$, $A' = \forall z [z/x] B'$, B' is an alphabetic variant of B, and either $z = x$ or $z \notin \mathit{Var}(B')$.
5. $A = \langle y : x \rangle B$, $A' = \langle y : z \rangle [z/x] B'$, B' is an alphabetic variant of B, and either $z = x$ or $z \notin \mathit{Var}(B')$.
6. $A = \Box B$, $A' = \Box B'$, and B' is an alphabetic variant of A'.

LEMMA 3.8 (ALPHA-CONVERSION LEMMA)
If a formula A' is an alphabetic variant of a formula A, then for any world w in any structure $\mathcal{S}$ and any interpretation V on $\mathcal{S}$,

$$w, V \Vdash_{\mathcal{S}} A \text{ iff } w, V \Vdash_{\mathcal{S}} A'.$$

PROOF by induction on A.

1. A is atomic. Then $A = A'$ and the claim is trivial.
2. A is $\neg B$. Then A' is $\neg B'$, where B' is an alphabetic variant of B. By induction hypothesis, $w, V \Vdash B$ iff $w, V \Vdash B'$. So $w, V \Vdash \neg B$ iff $w, V \Vdash \neg B'$ by definition 2.7.
3. A is $B \supset C$. Then A' is $B' \supset C'$, where B', C' are alphabetic variants of B, C, respectively. By induction hypothesis, $w, V \Vdash B$ iff $w, V \Vdash B'$ and $w, V \Vdash C$ iff $w, V \Vdash C'$. So $w, V \Vdash B \supset C$ iff $w, V \Vdash B' \supset C'$ by definition 2.7.
4. A is $\forall x B$. Then A' is either $\forall x B'$ or $\forall z [z/x] B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. If B is $\forall x B'$, then by definition 2.7, $w, V \Vdash \forall x B$ iff $w, V' \Vdash B$ for all existential x-variants V' of V on w, which, by induction hypothesis, holds iff $w, V' \Vdash B'$ for all such V', i.e. (by definition 2.7 again) iff $w, V \Vdash \forall x B'$.

   Consider then the case where B is $\forall z [z/x] B'$, with $z \notin \mathit{Var}(B')$. This means that z is not free in B, because alphabetic variants never differ in their free variables. Now if $w, V \nVdash \forall z [z/x] B'$, then by definition 2.7 there is an existential z-variant V^* of V on w such that $w, V^* \nVdash [z/x] B'$. And then $w, (V^*)^{[z/x]} \nVdash B'$ by lemma 3.6. By induction hypothesis, then $w, (V^*)^{[z/x]} \nVdash B$. Let V' be the x-variant of V on w with $V'(x) = (V^*)^{[z/x]}(x) = V^*(z)$. Since z is not free in B, V' and $(V^*)^{[z/x]}$ agree at w

about all free variables in B. So by the coincidence lemma 2.9, $w, V' \nVdash B$. And so $w, V \nVdash \forall x B$ by definition 2.7.

The converse, that if $w, V \Vdash \forall z[z/x]B'$, then $w, V \Vdash \forall x B$, follows from the fact that $\forall x B$ is $\forall x[x/z][z/x]B'$ and thus an alphabetic variant of $\forall z[z/x]B'$.

5. A is $\langle y : x\rangle B$. Then A' is either $\langle y : x\rangle B'$ or $\langle y : z\rangle[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that B is $\langle y : x\rangle B'$. By definition 3.2, $w, V \Vdash \langle y : x\rangle B$ iff $w, V' \Vdash B$ where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. By induction hypothesis, this holds iff $w, V' \Vdash B'$, i.e. (by definition 2.7 again) iff $w, V \Vdash \langle y : x\rangle B'$.

   Consider then the case where B is $\langle y : z\rangle[z/x]B'$, with $z \notin \mathit{Var}(B')$. This means that z is not free in B, because alphabetic variants never differ in their free variables. By definition 2.7, $w, V \Vdash \langle y : x\rangle B$ iff $w, V' \Vdash B$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. Let V^* be the z-variant of V on w with $V^*_w(z) = V'_w(x) = V_w(y)$. Since z is not free in B, V' and $(V^*)^{[z/x]}$ agree at w about all variables in B. So by the coincidence lemma 2.9, $w, V' \Vdash B$ iff $w, (V^*)^{[z/x]} \Vdash B$. By induction hypothesis, $w, (V^*)^{[z/x]} \Vdash B$ iff $w, (V^*)^{[z/x]} \Vdash B'$. By lemma 3.6, $w, (V^*)^{[z/x]} \Vdash B'$ iff $w, V^* \Vdash [z/x]B'$. And by definition 2.7, $w, V^* \Vdash [z/x]B'$ iff $w, V \Vdash \langle y : z\rangle[z/x]B'$.

6. A is $\Box B$. Then A' is $\Box B'$, where B' is an alphabetic variant of B. By definition 2.7, $w, V \Vdash \Box B$ iff $w', V' \Vdash B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$, and $w, V \Vdash \Box B'$ iff $w', V' \Vdash B'$ for all such w', V'. By induction hypothesis, $w', V' \Vdash B$ iff $w', V' \Vdash B'$. So $w, V \Vdash \Box B$ iff $w, V \Vdash \Box B'$ by definition 2.7. ■

Now for the more general version of lemma 3.6.

Lemma 3.9 (Restricted substitution lemma)
Let A be a sentence in a language $\mathcal{L}$ of quantified modal logic (with or without substitution), $\mathcal{S}$ a counterpart structure for $\mathcal{L}$, w a world in $\mathcal{S}$, V an interpretation on $\mathcal{S}$. Then

(i) $w, V^{[y/x]} \Vdash_{\mathcal{S}} A$ iff $w, V \Vdash_{\mathcal{S}} [y/x]A$, provided that either
   (a) y and x are modally separated in A, or
   (b) there is no world $w' \in W$ and counterpart relation $C \in K_{w,w'}$ such that $V_w(y)$ has multiple C-counterparts at w'.

(ii) if $w, V^{[y/x]} \Vdash_{\mathcal{S}} A$, then $w, V \Vdash_{\mathcal{S}} [y/x]A$, provided that y is modally free for x in A.

(Proviso (i).(b) is meant to include cases where $V_w(y)$ is undefined.)

Proof If y and x are the same variable, then $V^{[y/x]}$ is V, and $[y/x]A$ is A; so trivially $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash [y/x]A$. Assume then that y and x are different variables. The proof is by induction on A.

For the base case, the provisos can be ignored: $w, V^{[y/x]} \Vdash Px_1 \dots x_n$ iff $w, V \Vdash [y/x]Px_1 \dots x_n$ by the same reasoning as in lemma 3.6. For complex A, the induction hypothesis is that (i) and (ii) hold for formulas of lower complexity, in particular for subformulas of A. Note that if one of the provisos of (i) and (ii) applies to A, then it also applies to subformulas for A. (For example, if y is modally free for x in A, then y is modally free for x in every subformula of A.) Moreover, if A is not of the form $\Box B$, then the proviso of (ii) entails proviso (a) of (i), because y is modally free for x in $A \neq \Box B$ only if y and x do not occur together in the scope of a modal operator in A.

1. A is $\neg B$. By definition 2.7, $w, V^{[y/x]} \Vdash \neg B$ iff $w, V^{[y/x]} \nVdash B$. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B, by induction hypothesis, $w, V^{[y/x]} \nVdash B$ iff $w, V \nVdash [y/x]B$. And the latter holds iff $w, V \Vdash [y/x]\neg B$ by definitions 2.7 and 3.3.

2. A is $B \supset C$. By definition 2.7, $w, V^{[y/x]} \Vdash B \supset C$ iff $w, V^{[y/x]} \nVdash B$ or $w, V^{[y/x]} \Vdash C$. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B and C, by induction hypothesis, $w, V^{[y/x]} \nVdash B$ iff $w, V \nVdash [y/x]B$, and $w, V^{[y/x]} \Vdash C$ iff $w, V \Vdash [y/x]C$. So $w, V^{[y/x]} \Vdash B \supset C$ iff $w, V \Vdash [y/x](B \supset C)$ by definitions 2.7 and 3.3.

3. A is $\forall zB$. Assume first that $[y/x]\forall zB$ is $\forall[y/x]z[y/x]B$, i.e. (by definition 3.3) neither $z = y$ and $x \in \mathit{Varf}(B)$ nor $z = x$ and $y \in \mathit{Varf}(B)$. By definition 2.7, $w, V \Vdash \forall[y/x]z[y/x]B$ iff $w, V' \Vdash [y/x]B$ for all existential $[y/x]z$-variants V' of V on w. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B, by induction hypothesis, $w, V' \Vdash [y/x]B$ iff $w, V'^{[y/x]} \Vdash B$.

   Now assume $z \notin \{x, y\}$. Then $V_w'^{[y/x]}(x) = V_w'^{[y/x]}(y) = V_w'(y) = V_w(y)$ and $V_w'^{[y/x]}(z) = V_w'^{[y/x]}([y/x]z)$ is some arbitrary member of D_w. So the interpretations $V'^{[y/x]}$ coincide with the existential z-variants $V^{[y/x]\prime}$ of $V^{[y/x]}$ on w. Alternatively, if $z = x$, and thus $y \notin \mathit{Varf}(B)$, then $V_w'^{[y/x]}(x)$ is some arbitrary member of D_w, as is $V_w^{[y/x]\prime}(x)$. Similarly, if $z = y$ and thus $x \notin fvar(B)$, then $V_w'^{[y/x]}(y)$ is some arbitrary member of D_w, as is $V_w^{[y/x]\prime}(y)$. In either case, the interpretations $V_w'^{[y/x]}$ can be paired with the interpretations $V^{[y/x]\prime}$ such that the members of each pair agree at w about all free variables in B. So by the coincidence lemma 2.9, $w, V'^{[y/x]} \Vdash B$ for all existential $[y/x]z$-variants V' of V on w iff $w, V^{[y/x]\prime} \Vdash B$ for all existential z-variants $V^{[y/x]\prime}$ of $V^{[y/x]}$ on w, iff $w, V^{[y/x]} \Vdash \forall zB$ by definition 2.7.

   Second, assume $[y/x]\forall zB$ is $\forall v[y/x][v/z]B$, for some new variable v. By the α-conversion lemma 3.8, $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V^{[y/x]} \Vdash \forall v[v/z]B$. Since $v \notin \{x, y\}$, we can reason as above, with $[v/z]B$ in place of B, to show that $w, V \Vdash \forall v[y/x][v/z]B$ iff $w, V^{[y/x]} \Vdash \forall v[v/z]B$.

4. A is $\langle y_2 : z\rangle B$. This case is similar to the previous one. Assume first that $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : [y/x]z\rangle[y/x]B$, i.e. (by definition 3.3) neither $z = y$ and $x \in \mathit{Varf}(B)$ nor $z = x$ and $y \in \mathit{Varf}(B)$. By definition 2.7, $w, V \Vdash \langle [y/x]y_2 : [y/x]z\rangle[y/x]B$ iff $w, V' \Vdash [y/x]B$, where V' is the $[y/x]z$-variant of V on w with $V_w'([y/x]z) = V_w([y/x]y_2)$. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B, by induction hypothesis, $w, V' \Vdash [y/x]B$ iff $w, V'^{[y/x]} \Vdash B$.

Let V^* be the z-variant of $V^{[y/x]}$ on w with $V^*_w(z) = V_w([y/x]y_2)$. If $z \notin \{x, y\}$, then $V^*_w(x) = V^*_w(y) = V_w(y)$ and $V^*(z) = V_w([y/x]y_2)$. Moreover, $V'^{[y/x]}_w(x) = V'^{[y/x]}_w(y) = V_w(y)$ and $V'^{[y/x]}_w(z) = V'^{[y/x]}_w([y/x]z) = V_w([y/x]y_2)$. So $V'^{[y/x]}$ and V^* agree about all variables at w. Alternatively, if $z = x$, and thus $y \notin \mathit{Varf}(B)$, then $V'^{[y/x]}_w(x) = V_w([y/x]y_2) = V^*_w(x)$. Similarly, if $z = y$, and thus $x \notin \mathit{Varf}(B)$, then $V'^{[y/x]}_w(y) = V_w([y/x]y_2) = V^*_w(y)$. Either way, $V'^{[y/x]}$ and V^* agree at w about all free variables in B. By the coincidence lemma 2.9, $w, V'^{[y/x]} \Vdash B$ iff $w, V^* \Vdash B$, iff $w, V^{[y/x]} \Vdash \langle [y/x]y_2 : z\rangle B$ by definition 2.7.

5. A is $\Box B$. This is the interesting part. We have to go piecemeal.

   (i). By definition 2.7, $w, V^{[y/x]} \Vdash \Box B$ iff $w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ with wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. On the other hand, $w, V \Vdash [y/x]\Box B$ iff $w, V \Vdash \Box[y/x]B$ (by definition 3.3), iff $w', V' \Vdash [y/x]B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. Since the provisos of (i) carry over from $\Box B$ to B, by induction hypothesis, $w', V'^{[y/x]} \Vdash B$ iff $w', V' \Vdash [y/x]B$. So we have to show that

   (1) $\quad w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ such that wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$

   iff

   (2) $\quad w', V'^{[y/x]} \Vdash B$ for all w', V' such that wRw' and $V_w \triangleright V'_{w'}$.

   (1) implies (2) because every interpretation $V'^{[y/x]}$ with $V_w \triangleright V'_{w'}$ is also an interpretation $V^{[y/x]\prime}$ with $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. The converse, however, may fail: both $V'^{[y/x]}_{w'}$ and $V^{[y/x]\prime}_{w'}$ assign to x and y some counterpart of $V_w(y)$ (if there is any). But while $V'^{[y/x]}_{w'}$ assigns *the same* counterpart to x and y, $V^{[y/x]\prime}_{w'}$ may choose different counterparts for x and y relative to the same counterpart relation.

   If there is no counterpart relation relative to which $V_w(y)$ has multiple counterparts, then this cannot happen. Thus under proviso (b), each $V^{[y/x]\prime}$ with $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$ is also a $V'^{[y/x]}$ with $V'_w \triangleright V'^{[y/x]}_{w'}$, and so (2) implies (1).

   For proviso (a), assume x and y do not both occur in the scope of a modal operator in $\Box B$. Then either x or y does not occur at all in $\Box B$. Assume first that x does not occur in $\Box B$. Then $[y/x]\Box B$ is $\Box B$ (by definition 3.3), and $w, V^{[y/x]} \Vdash \Box B$ iff $w, V \Vdash [y/x]\Box B$ by the coincidence lemma 2.9. Alternatively, assume that y does not occur in $\Box B$, and that (2) holds. In order to derive (1), consider any w'-image $V^{[y/x]\prime}$ of $V^{[y/x]}$ at w; i.e. for some counterpart relation $C \in K_{w,w'}$ and all variables z, $V^{[y/x]\prime}_{w'}(z)$ is a C-counterpart of $V^{[y/x]}_w$ (or undefined if there is none). Let V^* be like $V^{[y/x]\prime}$ except that $V^*_{w'}(y) = V^{[y/x]\prime}_{w'}(x)$. Let V' be like V^* except that $V'_{w'}(x)$ is some C-counterpart of $V_w(x)$, or undefined if there is none. Then $V_w \triangleright V'_{w'}$. (In particular, $V'_{w'}(y) = V^*_{w'}(y) = V^{[y/x]\prime}(x)$ is some C-counterpart of $V^{[y/x]}_w(x) = V_w(y)$, or undefined if there is none.) So by (2), $w', V'^{[y/x]} \Vdash B$. But $V'^{[y/x]} = V^*$ (since $V^*_{w'}(x) = V^*_{w'}(y)$). So $w', V^* \Vdash B$. And since $y \notin \mathit{Var}(B)$ and V^* is a y-variant of $V^{[y/x]\prime}$ on w', by the coincidence lemma 2.9, $w', V^* \Vdash B$ iff $w', V^{[y/x]\prime} \Vdash B$.

   (ii). Assume $w, V^{[y/x]} \Vdash \Box B$. By definition 2.7, then $w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ with wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. As before, every interpretation $V'^{[y/x]}$ with $V_w \triangleright V'_{w'}$ is also an interpretation $V^{[y/x]\prime}$ with $V_w^{[y/x]} \triangleright V_{w'}^{[y/x]\prime}$. So $w', V'^{[y/x]} \Vdash B$ for all $w', V'^{[y/x]}$ with wRw' and $V_w \triangleright V'_{w'}$.

If y is modally free for x in $\Box B$, then y is modally free for x in B. Then by induction hypothesis, $w', V' \Vdash [y/x]B$ *if* $w', V'^{[y/x]} \Vdash B$. So $w', V' \Vdash [y/x]B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. By definition 2.7, this means that $w, V \Vdash \Box[y/x]B$, and so $w, V \Vdash [y/x]\Box B$ by definition 3.3. ■

The converse of (ii) is not true. E.g., $w, V \Vdash [y/x]\Box x{=}y$ does not imply $w, V^{[y/x]} \Vdash \Box x{=}y$. So the operation $[y/x]$, as defined in definition 3.3, does not always satisfy the "substitution lemma", not even when y is modally free for x.

Can we fix the definition? No – at least not if we allow for positive models. There is no operation Φ on sentences in standard languages of quantified modal logic such that in any (positive) model, $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash \Phi(A)$, and therefore no translation of $\langle y : x\rangle A$ into those languages. To prove this, we show that there are distinctions one can draw with $\langle y : x\rangle$ that cannot be drawn without it. In particular, the substitution quantifier allows us to say that an individual y has multiple counterparts at some accessible world (under the same counterpart relation): $\langle y : x\rangle\Diamond y{\neq}x$.

(In negative models, $\langle y : x\rangle A$ can be translated into $\exists x(x{=}y \wedge A) \vee (\neg Ey \wedge [y/x]A)$, which still has the downside of being very impractical, since $\Phi(A)$ can have much greater syntactic complexity than A.)

It is clear that $\Diamond y \neq y$ is not an adequate translation of $\langle y : x\rangle\Diamond x \neq y$. Before substituting y for x in $\Diamond x \neq y$, we would have to make x free for y by renaming the modally bound occurrence of y. However, the diamond, unlike the quantifier $\forall y$, binds y in such a way that the domain over which it ranges (the counterparts of y's original referent) depends on the previous reference of y. So we can't just replace y by some other variable z, translating $\langle y : x\rangle\Diamond x{\neq}y$ as $\Diamond y{\neq}z$. This only works if z happens to corefer with y. Since we can't presuppose that there is always another name available for any given individual, we would somehow have to introduce a name z that corefers with y. For instance, we could translate $\Diamond x{\neq}y$ into $\exists z(y{=}z \wedge \Diamond x{\neq}z)$. Now x has become free for y in the scope of the diamond, so we can translate $\langle y : x\rangle\Diamond x{\neq}y$ as $\exists z(y{=}z \wedge \Diamond x{\neq}y)$. The problem with this is that the quantifier $\exists$ ranges only over existing objects, while $\langle y : x\rangle$ bears no such restriction. In positive models, $V_w(y)$ can have multiple counterparts even if it lies outside D_w, so that $\exists z(y = z \wedge \Diamond x \neq y)$ is false. (One would need an "outer quantifier" in place of $\exists$.)

Here is the full proof.

Theorem 3.10 (Undefinability of substitution)
There is no operation Φ on formulas A in a standard language $\mathcal{L}$ of quantified modal logic such that for all worlds w in all positive counterpart models $\langle \mathcal{S}, V\rangle$, $w, V \Vdash_{\mathcal{S}} \Phi(A)$ iff $w, V^{[y/x]} \Vdash_{\mathcal{S}} A$.

Proof Let $\mathcal{M}_1 = \langle \mathcal{S}_1, V \rangle$ be a positive counterpart model with $W = \{w\}$, $R = \{\langle w, w \rangle\}$, $U_w = \{x, y, y^*\}$, $D_w = \{x\}$, $K_{w,w} = \{\{\langle d, d \rangle : d \in U_w\}\}, V_w(y) = y$, $V_w(z) = x$ for every variable $z \neq y$, and $V_w(P) = \emptyset$ for all non-logical predicates P. Let $\mathcal{M}_2 = \langle \mathcal{S}_2, V \rangle$ be like $\mathcal{M}_1$ except that y^* is also a counterpart of y, i.e. $K_{w,w'} = \{\{\langle x, x \rangle, \langle y, y \rangle, \langle y^*, y^* \rangle, \langle y, y^* \rangle\}\}$. Then $w, V^{[y/x]} \Vdash_{\mathcal{S}_2} \Diamond y \neq x$, but $w, V^{[y/x]} \nVdash_{\mathcal{S}_1} \Diamond y \neq x$.

On the other hand, every $\mathcal{L}$-sentence has the same truth-value at w under V in both models. We prove this by showing that for every $\mathcal{L}$-sentence A, $w, V \Vdash_{\mathcal{S}_1} A$ iff $w, V \Vdash_{\mathcal{S}_2} A$ iff $w, V^* \Vdash_{\mathcal{S}_2} A$, where V^* is the y-variant of V on w with $V^*_w(y) = V_w(y^*)$.

1. A is $Px_1 \ldots x_n$. It is clear that $w, V \Vdash_{\mathcal{S}_1} Px_1 \ldots x_n$ iff $w, V \Vdash_{\mathcal{S}_2} Px_1 \ldots x_n$ because counterpart relations do not figure in the evaluation of atomic formulas. Moreover, for non-logical P, $w, V \nVdash_{\mathcal{S}_2} Px_1 \ldots x_n$ and $w, V^* \nVdash_{\mathcal{S}_2} Px_1 \ldots x_n$, because $V_w(P) = V^*_w(P) = \emptyset$. For the identity predicate, observe that $w, V \nVdash_{\mathcal{S}_2} u = v$ iff exactly one of u, v is y, since $V_w(z) = x$ for all terms $z \neq y$. For the same reason, $w, V^* \nVdash_{\mathcal{S}_2} u = v$ iff exactly one of u, v is y. So $w, V \Vdash_{\mathcal{S}_2} u = v$ iff $w, V^* \Vdash_{\mathcal{S}_2} u = v$.

2. A is $\neg B$. $w, V \Vdash_{\mathcal{S}_1} \neg B$ iff $w, V \nVdash_{\mathcal{S}_1} B$ by definition 2.7, iff $w, V \nVdash_{\mathcal{S}_2} B$ by induction hypothesis, iff $w, V \Vdash_{\mathcal{S}_2} \neg B$ by definition 2.7. Moreover, $w, V \nVdash_{\mathcal{S}_2} B$ iff $w, V^* \nVdash_{\mathcal{S}_2} B$ by induction hypothesis, iff $w, V^* \Vdash_{\mathcal{S}_2} \neg B$ by definition 2.7.

3. A is $B \supset C$. $w, V \Vdash_{\mathcal{S}_1} B \supset C$ iff $w, V \nVdash_{\mathcal{S}_1} B$ or $w, V \Vdash_{\mathcal{S}_1} C$ by definition 2.7, iff $w, V \nVdash_{\mathcal{S}_2} B$ or $w, V \Vdash_{\mathcal{S}_2} C$ by induction hypothesis, iff $w, V \Vdash_{\mathcal{S}_2} B \supset C$ by definition 2.7. Moreover, $w, V \nVdash_{\mathcal{S}_2} B$ or $w, V \Vdash_{\mathcal{S}_2} C$, iff $w, V^* \nVdash_{\mathcal{S}_2} B$ or $w, V^* \Vdash_{\mathcal{S}_2} C$ by induction hypothesis, iff $w, V^* \Vdash_{\mathcal{S}_2} B \supset C$ by definition 2.7.

4. A is $\forall z B$. Let v be a variable not in $Var(B) \cup \{y\}$. By lemma 3.8, $w, V \Vdash_{\mathcal{S}_1} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_1} \forall v [v/z] B$. By definition 2.7, $w, V \Vdash_{\mathcal{S}_1} \forall v [v/z] B$ iff $w, V' \Vdash_{\mathcal{S}_1} [v/z] B$ for all existential v-variants V' of V on w. As $D_w = \{x\}$ and $V(v) = x$, the only such v-variant is V itself. So $w, V \Vdash_{\mathcal{S}_1} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_1} [v/z] B$. By the same reasoning, $w, V \Vdash_{\mathcal{S}_2} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_2} [v/z] B$. But by induction hypothesis, $w, V \Vdash_{\mathcal{S}_1} [v/z] B$ iff $w, V \Vdash_{\mathcal{S}_2} [v/z] B$. So $w, V \Vdash_{\mathcal{S}_1} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_2} \forall z B$. Moreover, by induction hypothesis, $w, V \Vdash_{\mathcal{S}_2} [v/z] B$ iff $w, V^* \Vdash_{\mathcal{S}_2} [v/z] B$, iff $w, V^* \Vdash_{\mathcal{S}_2} \forall v [v/z] B$ because V^* is the only existential v-variant of V^* on w, iff $w, V^* \Vdash_{\mathcal{S}_2} \forall z B$ by lemma 3.8.

5. A is $\Box B$. In both structures, the only world accessible from w is w itself. Also in $\mathcal{S}_1$, V is the only w-image of V at w. So by definition 2.7, $w, V \Vdash_{\mathcal{S}_1} \Box B$ iff $w, V \Vdash_{\mathcal{S}_1} B$. In $\mathcal{S}_2$, there are two w-images of V at w: V and V^*. So $w, V \Vdash_{\mathcal{S}_2} \Box B$ iff both $w, V \Vdash_{\mathcal{S}_2} B$ and $w, V^* \Vdash_{\mathcal{S}_2} B$. By induction hypothesis, $w, V \Vdash_{\mathcal{S}_1} B$ iff both $w, V \Vdash_{\mathcal{S}_2} B$ and $w, V^* \Vdash_{\mathcal{S}_2} B$. So $w, V \Vdash_{\mathcal{S}_1} \Box B$ iff $w, V \Vdash_{\mathcal{S}_2} \Box B$. Moreover, in $\mathcal{S}_2$, V^* is the only w-image of V^* at w. So $w, V^* \Vdash_{\mathcal{S}_2} \Box B$ iff $w, V^* \Vdash_{\mathcal{S}_2} B$. By induction hypothesis, $w, V^* \Vdash_{\mathcal{S}_2} B$ iff $w, V \Vdash_{\mathcal{S}_2} B$. So $w, V^* \Vdash_{\mathcal{S}_2} \Box B$ iff both $w, V^* \Vdash_{\mathcal{S}_2} B$ and $w, V \Vdash_{\mathcal{S}_2} B$, which as we just saw holds iff $w, V \Vdash_{\mathcal{S}_2} \Box B$. ■

What we can do instead is introduce a new syntactic construction into the language that satisfies the substitution lemma by stipulation. This is what the substitution quantifier does. I have given its semantics in definition 3.2 by saying that $w, V \Vdash \langle y : x \rangle A$ iff

$w, V' \Vdash A$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. By the locality lemma (corollary 2.10 of lemma 2.9), it immediately follows that

$$w, V \Vdash \langle y : x \rangle A \text{ iff } w, V^{[y/x]} \Vdash A.$$

In the following, I will consider both systems in extended languages that include the substitution quantifier $\langle y : x \rangle$ and systems in standard languages that exclude it. The advantage of having the substitution quantifier is that it not only adds welcome expressive resources to the language, but also makes the logic and model theory somewhat more streamlined, because the corresponding versions of principles like Leibniz's Law,

$$x = y \supset (A \supset \langle y : x \rangle A)$$

hold without restrictions.

It will be useful to have a notion of substitution that applies to several variables at once. To this end, let's generalise definition 3.3 (classical substitution).

Definition 3.11 (Classical substitution, generalised)
A *substitution* on a language $\mathcal{L}$ is a total function $\sigma : \mathit{Var}(\mathcal{L}) \to \mathit{Var}(\mathcal{L})$. If σ is injective, it is called a *transformation*. I write $[y_1, \ldots, y_n / x_1, \ldots, x_n]$ for the substitution that maps x_1 to y_1, $\ldots$, x_n to y_n, and every other variable to itself.

Application of a substitution σ to a formula A is defined as follows.

$$\begin{aligned}
\sigma(Px_1 \ldots x_n) &= P\sigma(x_1) \ldots \sigma(x_n) \\
\sigma(\neg A) &= \neg\sigma(A); \\
\sigma(A \supset B) &= \sigma(A) \supset \sigma(B); \\
\sigma(\forall z A) &= \begin{cases} \forall v \sigma'([v/z]A) & \text{if there is an } x \text{ free in } \forall z A \text{ with } \sigma(x) = \sigma(z), \\ \forall \sigma(z) \sigma(A) & \text{otherwise,} \end{cases} \\
&\quad \text{where } \sigma' \text{ is like } \sigma \text{ except that } \sigma'(v) = v, \text{ and } v \text{ is the alphabetically} \\
&\quad \text{first variable not in } \sigma(A); \\
\sigma(\langle y_2 : z \rangle A) &= \begin{cases} \langle \sigma'(y_2) : v \rangle \sigma([v/z]A) & \text{if there is an } x \neq v \text{ in } \mathit{Varf}(A) \text{ with } \sigma(x) = \sigma(z), \\ \langle \sigma(y_2) : \sigma(z) \rangle \sigma(A) & \text{otherwise,} \end{cases} \\
&\quad \text{where } \sigma' \text{ is like } \sigma \text{ except that } \sigma'(v) = v, \text{ and } v \text{ is the alphabetically} \\
&\quad \text{first variable not in } \sigma(A); \\
\sigma(\Box A) &= \Box\sigma(A).
\end{aligned}$$

I will also write σA or A^σ instead of $\sigma(A)$. If Γ is a set of formulas, I write $\sigma(\Gamma)$ or Γ^σ for $\{C^\tau : C \in \Gamma\}$.

Here is the corresponding generalisation of $V^{[y/x]}$.

> DEFINITION 3.12 (INTERPRETATION UNDER SUBSTITUTION, GENERALISED)
> For any interpretation V on a structure $\mathcal{S}$ and substitution σ, V^σ is the interpretation that is like V except that for any world w in $\mathcal{S}$ and variable x, $V^\sigma_w(x) = V_w(\sigma(x))$.

Substitutions can be composed. If σ and τ are substitutions, then $\tau\cdot\sigma$ is the substitution that maps each variable x to $\tau(\sigma(x))$. Observe that composition behaves differently in superscripts of formulas than in superscripts of interpretations: for formulas A,

$$(A^\sigma)^\tau = \tau(\sigma(A)) = A^{\tau\cdot\sigma},$$

but for interpretations V,

$$(V^\sigma)^\tau = V^{\sigma\cdot\tau}.$$

That's because $(V^\sigma)^\tau_w(x) = V^\sigma_w(\tau(x)) = V_w(\sigma(\tau(x)) = V_w(\sigma\cdot\tau(x)) = V^{\sigma\cdot\tau}_w(x)$.

Definition 3.11 draws attention to the class of injective substitutions, or transformations. A transformation never substitutes two distinct variables by the same variable. For instance, the identity substitution $[x/x]$ or the swapping operation $[x, y/y, x]$ are transformations. What's special about such substitutions is that they make capturing impossible: for the free variable y in $\forall x A(y)$ to be captured by the initial quantifier $\forall x$ after substitution, x and y have to be replaced by the same variable. Correspondingly, definition 3.11 entails that if σ is a transformation, then $\sigma(A)$ is simply A with all variables simultaneously replaced by their σ-value. Transformations satisfy the substitution lemma without any restrictions, even for modal formulas.

> LEMMA 3.13 (TRANSFORMATION LEMMA)
> For any world w in any structure $\mathcal{S}$, any interpretation V on $\mathcal{S}$, any formula A and transformation τ, $w, V^\tau \Vdash A$ iff $w, V \Vdash A^\tau$.

PROOF by induction on A.

1. $A = Px_1 \ldots x_n$. $w, V^\tau \Vdash Px_1 \ldots x_n$ iff $\langle V^\tau_w(x_1), \ldots, V^\tau_w(x_n)\rangle \in V^\tau_w(P)$, iff $\langle V_w(x^\tau_1), \ldots, V_w(x^\tau_n)\rangle \in V_w(P)$, iff $w, V \Vdash (Px_1 \ldots x_n)^\tau$.
2. $A = \neg B$. $w, V^\tau \Vdash \neg B$ iff $w, V^\tau \nVdash B$, iff $w, V \nVdash B^\tau$ by induction hypothesis, iff $w, V \Vdash (\neg B)^\tau$.

3. $A = B \supset C$. $w, V^\tau \Vdash B \supset C$ iff $w, V^\tau \nVdash B$ or $w, V^\tau \Vdash C$, iff $w, V \nVdash B^\tau$ or $w, V \Vdash C^\tau$ by induction hypothesis, iff $w, V \Vdash (B \supset C)^\tau$.

4. $A = \langle y : x \rangle B$. By definition 3.2, $w, V^\tau \Vdash \langle y : x \rangle B$ iff $w, (V^\tau)^{[y/x]} \Vdash B$. Now $(V^\tau)^{[y/x]}_w(x) = V^\tau_w(y) = V_w(y^\tau) = V^{[y^\tau/x^\tau]}_w(x^\tau) = (V^{[y^\tau/x^\tau]})^\tau_w(x)$. And for any variable $z \neq x$, $(V^\tau)^{[y/x]}_w(z) = V^\tau_w(z) = V_w(z^\tau) = V^{[y^\tau/x^\tau]}_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= (V^{[y^\tau/x^\tau]})^\tau_w(z)$. So $(V^\tau)^{[y/x]}$ coincides with $(V^{[y^\tau/x^\tau]})^\tau$ at w. By the locality lemma 2.10, $w, (V^\tau)^{[y/x]} \Vdash B$ iff $w, (V^{[y^\tau/x^\tau]})^\tau \Vdash B$. By induction hypothesis, the latter holds iff $w, V^{[y^\tau/x^\tau]} \Vdash B^\tau$, iff $w, V \Vdash \langle y^\tau : x^\tau \rangle B^\tau$ by definition 3.2, iff $w, V \Vdash (\langle y : x \rangle B)^\tau$ by definition 3.11.

5. $A = \forall x B$. Assume $w, V^\tau \nVdash \forall x B$. Then $w, V^* \nVdash B$ for some existential x-variant V^* of V^τ on w. Let V' be the (existential) x^τ-variant of V on w with $V'_w(x^\tau) = V^*_w(x)$. Then ${V'}^\tau_w(x) = V^*_w(x)$, and for any variable $z \neq x$, ${V'}^\tau_w(z) = V'_w(z^\tau) = V_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= V^\tau_w(z) = V^*_w(z)$. So ${V'}^\tau$ coincides with V^* on w, and by locality (lemma 2.10), $w, {V'}^\tau \nVdash B$. By induction hypothesis, then $w, V' \nVdash B^\tau$. So there is an existential x^τ-variant V' of V on w such that $w, V' \nVdash B^\tau$. By definition 2.7, this means that $w, V \nVdash \forall x^\tau B^\tau$, and hence $w, V \nVdash (\forall x B)^\tau$ by definition 3.11.

   In the other direction, assume $w, V \nVdash (\forall x B)^\tau$, and thus $w, V \nVdash \forall x^\tau B^\tau$. Then $w, V' \nVdash B^\tau$ for some existential x^τ-variant V' of V on w, and by induction hypothesis $w, {V'}^\tau \nVdash B$. Let V^* be the (existential) x-variant of V^τ on w with $V^*_w(x) = V'_w(x^\tau)$. Then $V^*_w(x) = {V'}^\tau_w(x)$, and for any variable $z \neq x$, $V^*_w(z) = V^\tau_w(z) = V_w(z^\tau) = V'_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= {V'}^\tau_w(z)$. So V^* coincides with ${V'}^\tau$ on w, and by locality (lemma 2.10), $w, V^* \nVdash B$. So there is an existential x-variant V^* of V^τ on w such that $w, V^* \nVdash B$. By definition 2.7, this means that $w, V^\tau \nVdash \forall x B$.

6. $A = \Box B$. Assume $w, V \nVdash \Box B^\tau$. Then $w', V' \nVdash B^\tau$ for some w', V' with wRw' and V' a w' image of V at w. This means that there is a counterpart relation $C \in K_{w,w'}$ such that for all variables x, $V'_{w'}(x)$ is some C-counterpart at w' of $V_w(x)$ at w (if any, else undefined). By induction hypothesis, $w', {V'}^\tau \nVdash B$. Since for all x, ${V'}^\tau_{w'}(x) = V'_{w'}(x^\tau)$ and $V^\tau_w(x) = V_w(x^\tau)$, it follows that ${V'}^\tau_{w'}(x)$ is a C-counterpart of $V^\tau_w(x)$ (if any, else undefined). So ${V'}^\tau$ is a w'-image of V^τ at w. Hence $w', {V'}^\tau \nVdash B$ for some $w', {V'}^\tau$ with wRw' and ${V'}^\tau$ a w'-image of V^τ at w. So $w, V^\tau \nVdash \Box B$.

   In the other direction, assume $w, V^\tau \nVdash \Box B$. Then $w', V^* \nVdash B$ for some w', V^* with wRw' and V^* a w' image of V^τ at w. This means that there is a counterpart relation $C \in K_{w,w'}$ such that for all variables x, $V^*_{w'}(x)$ is some C-counterpart at w' of $V^\tau_w(x)$ at w (if any, else undefined). Let V' be like V except that for all variables x, $V'_{w'}(x^\tau) = V^*_{w'}(x)$, and for all $x \notin \mathrm{Ran}(\tau)$, $V'_{w'}(x)$ is an arbitrary C-counterpart of $V_w(x)$, or undefined if there is none. V' is a w' image of V at w. Moreover, V^* is ${V'}^\tau$. By induction hypothesis, $w', V' \nVdash B^\tau$. So $w', V' \nVdash B^\tau$ for some w', V' with wRw' and V' a w' image of V at w. So $w, V \nVdash (\Box B)^\tau$. ■

For the substitution quantifier, we could introduce primitive polyadic quantifiers like $\langle y_1, y_2 : x_1, x_2 \rangle$, which says '$y_1$ is an x_1 and y_2 an x_2 such that', and stipulate that

$$w, V \Vdash \langle y_1, y_2 : x_1, x_2 \rangle A \text{ iff } w, V^{[y_1, y_2 / x_1, x_2]} \Vdash A.$$

Geach's $\langle x : y, z\rangle$ is then equivalent to $\langle x, x : y, z\rangle$. But it turns out that $\langle y_1, y_2 : x_1, x_2\rangle$ is definable.

We can't simply say that $\langle y_1, y_2 : x_1, x_2\rangle A$ is $\langle y_1 : x_1\rangle\langle y_2 : x_2\rangle A$, since the bound variable x_1 might capture y_2, as in the swapping operator $\langle x, y : y, x\rangle$. We must store the original value of y_2 in a temporary variable z: $\langle y_2 : z\rangle\langle y_1 : x_1\rangle\langle z : x_2\rangle$.

Definition 3.14 (Substitution sequences)
For any $n > 1$, sentence A and variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct, let $\langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$ abbreviate $\langle y_n : z\rangle\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z : x_n\rangle A$, where z is the alphabetically first variable not in A or $x_1, \ldots, x_n$.

Lemma 3.15 (Substitution sequence semantics)
For any world w in any structure $\mathcal{S}$, any interpretation V on $\mathcal{S}$,

$$w, V \Vdash_{\mathcal{S}} \langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A \text{ iff } w, V^{[y_1,\ldots,y_n/x_1,\ldots,x_n]} \Vdash_{\mathcal{S}} A.$$

Proof By definition 3.14, $w, V \Vdash \langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$ iff $w, V \Vdash \langle y_n : z\rangle\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z : x_n\rangle A$, for some z not in $x_1, \ldots, x_{n-1}, A$. By definition 3.2, $w, V \Vdash \langle y_n : z\rangle\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z : x_n\rangle A$ iff $w, V^{[y_n/z]} \Vdash \langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z : x_n\rangle A$, which by induction hypothesis holds iff $w, V^{[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]} \Vdash \langle z : x_n\rangle A$. By definition 3.2 again, $w, V^{[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]} \Vdash \langle z : x_n\rangle A$ iff $w, V^{[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]\cdot[z/x_n]} \Vdash A$. Now $[y_n/z]\cdot[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}]\cdot[z/x_n]$ is the function $\sigma : Var \rightarrow Var$ such that

$$\sigma(x) = [y_n/z]([y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}]([z/x_n](x))).$$

Since $z \notin x_1, \ldots, x_{n-1}$, this means that

$$\begin{aligned}
&\sigma(x_n) = y_n,\\
&\sigma(x_i) = y_i \text{ for } x_i \in \{x_1, \ldots, x_{n-1}\},\\
&\sigma(z) = y_n,
\end{aligned}$$

and $\sigma(x) = x$ for every other variable x. Since $z \notin \mathit{Var}(A)$, V^σ agrees at w with $V^{[y_1,\ldots,y_n/x_1,\ldots,x_n]}$ about all variables in A. So by the coincidence lemma 2.9, $w, V^\sigma \Vdash A$ iff $w, V^{[y_1,\ldots,y_n/x_1,\ldots,x_n]} \Vdash A$. ∎

4 Logics

I now want to describe the minimal logics that are characterised by our semantics. Following tradition, a *logic* (or *system*) in this context is simply a set of formulas, and I will describe such sets by recursive clauses corresponding to the axioms and rules of a Hilbert-style calculus.

Recall that we have two kinds of models: positive models with two domains, and negative models with a single domain. Accordingly we have two kinds of logics.

The logic of all positive models is essentially the combination of standard positive free logic with the propositional modal logic K. The only place to be careful is with substitution principles like Leibniz' Law, which either have to be expressed with object-language substitution or restricted as explained in the previous section. (If we add the unrestricted principles, we get logics for functional structures, as we'll see later.)

For a given first-order language language $\mathcal{L}$, standard positive free logic can be characterized as the smallest set L of formulas that contains

(Taut) all propositional tautologies in $\mathcal{L}$,

as well as all $\mathcal{L}$-instances of

(UD) $\forall xA \supset (\forall x(A \supset B) \supset \forall xB)$,

(VQ) $A \supset \forall xA$, provided x is not free in A,

(FUI) $\forall xA \supset (Ey \supset [y/x]A)$,

(∀Ex) $\forall xEx$,

(=R) $x = x$,

(LL) $x{=}y \supset A \supset [y/x]A$,

and that is closed under modus ponens, universal generalisation, and variable substitution:

(MP) if $\vdash_L A$ and $\vdash_L A \supset B$, then $\vdash_L B$,

(UG) if $\vdash_L A$, then $\vdash_L \forall xA$,

(Sub) if $\vdash_L A$, then $\vdash_L [y/x]A$.

Here, as always, $\vdash_L A$ means $A \in L$.[4]

In the logic of positive counterpart structures (without object-language substitution), (LL), (FUI) and (Sub) are restricted to cases where y is modally free for x in A:

(FUI*) $\forall xA \supset (Ey \supset [y/x]A)$, provided y is modally free for x in A,

(LL*) $x{=}y \supset A \supset [y/x]A$, provided y is modally free for x in A,

(Sub*) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is modally free for x in A.

4 [Fitting and Mendelsohn 1998] use $\forall xA \leftrightarrow A$ in place of (VQ), which precludes empty inner domains (as [Kutz 2000: 38] points out). The systems presented here do not validate the claim that something exists. To rule out empty inner domains in positive and negative models, $\exists xEx$ would be needed as extra axiom.

In addition, we include the modal schema

(K) $\Box A \supset (\Box(A \supset B) \supset \Box B)$,

and closure under necessitation,

(Nec) if $\vdash_L A$, then $\vdash_L \Box A$.

> Definition 4.1 (Minimal positive (quantified modal) logic)
> Given a standard language $\mathcal{L}$, the *minimal positive (quantified modal) logic* P in $\mathcal{L}$ is the smallest set $L \subseteq \mathcal{L}$ that contains all $\mathcal{L}$-instances of (Taut), (UD), (VQ), (FUI*), ($\forall$Ex), (=R), (LL*), (K), and that is closed under (MP), (UG), (Nec) and (Sub*).

We shall also be interested in stronger logics adequate for various classes of counterpart structures. As a first stab, I will adopt the following definition.

> Definition 4.2 (Positive logics)
> Given a standard language $\mathcal{L}$, a *positive (quantified modal) logic* in $\mathcal{L}$ is a an extension $L \supseteq \mathsf{P}$ of the minimal positive logic P in $\mathcal{L}$ such that L is closed under (MP), (UG), (Nec) and (Sub*).

As it stands, definition 4.2 allows for "logics" in which (say) F_1x is a theorem but not F_2x. This clashes with the idea that logical truths should be independent of the interpretation of non-logical terms. A more adequate definition would add a second-order closure condition to the effect that, roughly, whenever $\vdash_L A$ then $\vdash_L [B/Px_1 \ldots x_n]A$, where $[B/Px_1 \ldots x_n]A$ is A with all occurrences of the atomic formula $Px_1 \ldots x_n$ replaced by (the arbitrary formula) B. Making this precise requires some care, especially once we look at negative logics where $Fx \supset Ex$ is valid, but $\neg Fx \supset Ex$ is not. Definition 4.2 will do as long as we start off with a class of counterpart structures and look for a corresponding positive logic; that logic will always satisfy the definition 4.2.

The first-order closure condition (Sub*) at least excludes logics in which, for example, Fx is a theorem but not Fy. To see why (Sub*) needs the proviso, note that we could otherwise move from the (FUI*) instance $\vdash_L \forall x \Diamond Gxy \supset \Diamond Gzy$ to $\vdash_L \forall x \Diamond Gx \supset \Diamond Gyy$, which is invalid as long as individuals can have multiple counterparts.

You may also wonder why (Sub*) is needed in definition 4.1, since that the axioms are given by schemas: doesn't this mean that every substitution instance of an axiom is itself an axiom, and isn't this property of closure under substitution preserved by (MP), (UG) and (Nec)? Not quite. For example,

(6) $\quad v = y \supset (v = z \supset y = x)$

is an instance of (LL*), and

(7) $\quad x = y \supset (x = x \supset y = x)$

follows from (6) by (Sub*), but (7) is not itself an instance of (LL*). Of course it is possible to axiomatise P without (Sub*), and nothing really hangs on it. I have chosen the above axiomatisation just because I find it comparatively intuitive and convenient for the purposes of this paper.

> Theorem 4.3 (Soundness of P)
> Every member of P is valid in every positive counterpart model.

Proof We show that all P axioms are valid in every positive model, and that validity is closed under (MP), (UG), (Nec) and (Sub*).

1. (Taut). Propositional tautologies are valid in every model by the standard satisfaction rules for the connectives.
2. (UD). Assume $w, V \Vdash \forall x(A \supset B)$ and $w, V \Vdash \forall x A$ in some model. By definition 2.7, then $w, V' \Vdash A \supset B$ and $w, V' \Vdash A$ for every existential x-variant V' of V on w, and so $w, V' \Vdash B$ for every such V'. Hence $w, V \Vdash \forall x B$.
3. (VQ). Suppose $w, V \nVdash A \supset \forall x A$ in some model. Then $w, V \Vdash A$ and $w, V \nVdash \forall x A$. If x is not free in A, then by the coincidence lemma 2.9, $w, V' \Vdash A$ for every x-variant V' of V on D_w; so $w, V \Vdash \forall x A$. Contradiction. So if x is not free in A, then $A \supset \forall x A$ is valid in every model.
4. (FUI*). Assume $w, V \Vdash \forall x A$ and $w, V \Vdash Ey$ in some model. By definition 2.7, then $w, V' \Vdash A$ for all existential x-variants V' of V on w. So in particular, $w, V^{[y:x]} \Vdash A$. If y is modally free for x in A, then by lemma 3.9, $w, V \Vdash [y/x]A$.
5. ($\forall$Ex). By definition 2.7, $w, V \Vdash \forall x E x$ iff $w, V' \Vdash Ex$ for all existential x-variants V' of V on w, iff for all existential x-variants V' of V on w there is an existential y-variant V'' of V' on w such that $w, V'' \Vdash x = y$. But this is always the case: for any V', let V'' be $V'^{[x/y]}$.
6. (=R). By definition 2.3, $V_w(=) = \{\langle d, d\rangle : d \in U_w\}$, and by definition 2.3, $V_w(x) \in U_w$ in every positive model. So $w, V \Vdash x = x$ in every such model, by definition 2.7.
7. (LL*). Assume $w, V \Vdash x = y$, $w, V \Vdash A$, and y is modally free for x in A. Since $V_w(x) = V_w(y)$, V coincides with $V^{[y/x]}$ at w. So $w, V^{[y/x]} \Vdash A$ by the coincidence lemma 2.9. By lemma 3.9, $w, V^{[y/x]} \Vdash A$ only if $w, V \Vdash [y/x]A$. So $w, V \Vdash [y/x]A$.

8. (K). Assume $w, V \Vdash \Box(A \supset B)$ and $w, V \Vdash \Box A$. Then $w', V' \Vdash A \supset B$ and $w', V' \Vdash A$ for every w', V' such that wRw' and V' is a w'-image of V at w. Then $w', V' \Vdash B$ for any such w', V', and so $w, V \Vdash \Box B$.

9. (MP). Assume $w, V \Vdash A \supset B$ and $w, V \Vdash A$ in some model. By definition 2.7, then $w, V \Vdash B$ as well. So for any world w in any model, (MP) preserves truth at w.

10. (UG). Assume $w, V \nVdash \forall x A$ in some model $\mathcal{M}$. Then $w, V' \nVdash A$ for some existential x-variant V' of V on w. So A is invalid in a model like $\mathcal{M}$ but with V' as the interpretation function in place of V. Hence if A is valid in all positive models, then so is $\forall x A$.

11. (Nec). Assume $w, V \nVdash_{\mathcal{M}} \Box A$ in some model $\mathcal{M}$. Then $w', V' \nVdash A$ for some w' with wRw' and V' some w'-image of V at w. Let $\mathcal{M}^*$ be like $\mathcal{M}$ except with V' in place of V. $\mathcal{M}^*$ is a positive model. Since A is not valid in $\mathcal{M}^*$, it follows contrapositively that whenever A is valid in all positive models, then so is $\Box A$.

12. (Sub*). Assume $w, V \nVdash [y/x]A$ in some model $\langle \mathcal{S}, V \rangle$, and y is modally free for x in A. By lemma 3.9, then $w, V^{[y/x]} \nVdash A$. So A is invalid in the model $\langle \mathcal{S}, V^{[y/x]} \rangle$. Hence if A is valid in all positive models, then so is $[y/x]A$. ■

Let's move on to negative logics. Standard negative free logic replaces (=R) and (∀Ex) by

(∀=R) $\forall x(x = x)$,

(Neg) $Px_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n$.

In our single-domain models, we need two further axioms, as mentioned on p. 4:

(NA) $\neg Ex \supset \Box \neg Ex$,

(TE) $x = y \supset \Box(Ex \supset Ey)$.

As I said in section 2, (NA) should not be confused with the claim that no individual exists at any world that isn't a counterpart of something at the centre, which would require something like the Barcan Formula,

(BF) $\forall x \Box A \supset \Box \forall x A$.

But this isn't valid in the class of negative models. For example, if $W = \{w, w'\}$, wRw', $D_w = \emptyset$ and $D_{w'} = \{0\}$, then $w, V \Vdash \forall x \Box x \neq x$, but $w, V \nVdash \Box \forall x\, x \neq x$.

Definition 4.4 (Minimal (strongly) negative (quantified modal) logic) Given a standard language $\mathcal{L}$, the *minimal (strongly) negative (quantified modal) logic* N in $\mathcal{L}$ is the smallest set $L \subseteq \mathcal{L}$ that contains all $\mathcal{L}$-instances of (Taut), (UD), (VQ), (FUI*), (Neg), (LL*), (∀=R), (K), (NA), (TE), and that is closed under (MP), (UG), (Nec) and (Sub*).

Definition 4.5 (Negative logics)
Given a standard language $\mathcal{L}$, a *negative (quantified modal) logic* in $\mathcal{L}$ is a an extension $L \supseteq \mathsf{N}$ of the minimal negative logic N in $\mathcal{L}$ such that L is closed under (MP), (UG), (Nec) and (Sub*).

Theorem 4.6 (Soundness of N)
Every member of N is valid in every negative counterpart model.

Proof We show that all N axioms are valid in every negative model, and that validity is closed under (MP), (UG), (Nec) and (Sub*). The proofs for (Taut), (UD), (VQ), (FUI*), (LL*), (K), (MP), (UG), (Nec) and (Sub*) are just as in theorem 4.3. The remaining cases are

1. (Neg). Assume $w, V \Vdash Px_1 \dots x_n$ in some model. By definition 2.3, $V_w(P) \subseteq U_w^n$, and by definition 2.4, $U_w = D_w$ in negative models. So $V_w(P) \subseteq D_w^n$. By definition 2.7, $w, V \Vdash Px_1 \dots x_n$ therefore entails that $V_w(x_i) \in D_w$ for all $x_i \in x_1, \dots, x_n$, and that $w, V \Vdash Ex_i$ for all such x_i.
2. ($\forall$=R). $w, V \Vdash \forall x(x = x)$ iff $w, V' \Vdash x = x$ for all existential x-variants V' of V on w. This is always the case, since by definition 2.3 and 2.3, $V_w(=) = \{\langle d, d\rangle : d \in D_w\}$ in negative models.
3. (NA). Assume $w, V \Vdash \neg Ex$. By definition 2.7, this means that $V_w(x) \notin D_w$, and therefore that $V_w(x)$ is undefined if the model is negative. But if $V_w(x)$ is undefined, then there is no world w', individual d and counterpart relation $C \in K_{w,w'}$ such that $\langle V_w(x), w\rangle C \langle d, w'\rangle$. By definitions 2.7 and 2.6, it follows that there is no world w' and interpretation V' with wRw' and $V_w \triangleright V'_{w'}$ such that $w', V' \Vdash Ex$. So then $w, V \Vdash \Box\neg Ex$ by definition 2.7. Thus $w, V \Vdash \neg Ex \supset \Box\neg Ex$.
4. (TE). Assume $w, V \Vdash x = y$. Then $V_w(x) = V_w(y)$ by definitions 2.3 and 2.7. Let w', V' be such that wRw' and $V_w \triangleright V'_{w'}$, and $w', V' \Vdash Ex$. By definition 2.7, the latter means that $V'_{w'}(x)$ is some member of D_w. Moreover, $V_w \triangleright V'_{w'}$ means that there is a $C \in K_{w,w'}$ such that this $V'_{w'}(x) \in D_w$ is a C-counterpart of $V_w(x)$. It follows that $V_w(y) = V_w(x)$ has at least one C-counterpart at w', so $V'_{w'}(y)$ must be some such counterpart, which can only be in D_w. So $w', V' \Vdash Ey$. So if $w, V \Vdash x = y$, then $w, V \Vdash \Box(Ex \supset Ey)$, by definition 2.7, and so $w, V \Vdash x = y \supset \Box(Ex \supset Ey)$. ■

In the remainder of this section, I will prove a few properties derivable from the above axiomatisations. (Some of these will be needed later on in the completeness proof.) To this end, let $\mathcal{L}$ range over standard languages of quantified modal logic, and L over an arbitrary positive or negative logic in $\mathcal{L}$.

Lemma 4.7 (Closure under propositional consequence)
For all $\mathcal{L}$-formulas $A_1, \dots, A_n, B$,

(PC) if $\vdash_L A_1, \dots, \vdash_L A_n$, and B is a propositional consequence of $A_1, \dots, A_n$, then $\vdash_L B$.

Proof If B is a propositional consequence of $A_1, \dots, A_n$, then $A_1 \supset (\dots \supset (A_n \supset B)\dots)$ is a tautology. So by (Taut), $\vdash_L A_1 \supset (\dots \supset (A_n \supset B)\dots)$. If $\vdash_L A_1, \dots, \vdash_L A_n$, then by n applications of (MP), $\vdash_L B$. ■

When giving proofs, I will often omit reference to (PC).

Lemma 4.8 (Redundant axioms)
For any $\mathcal{L}$-formulas A and variables x,

(∀Ex) $\vdash_L \forall x Ex$,

(∀=R) $\vdash_L \forall x(x = x)$.

Proof If L is positive, then (∀Ex) is an axiom. In N, we have $\vdash_L x = x \supset Ex$ by (Neg); so by (UG) and (UD), $\vdash_L \forall x(x = x) \supset \forall x Ex$. Since $\vdash_L \forall x(x = x)$ by (=R), $\vdash_L \forall x Ex$.

If L is negative, then (∀=R) is an axiom. In P, we have $\vdash_L x = x$ by (=R), and so (∀=R) by (UG). ■

Lemma 4.9 (Existence and Self-Identity)
If L is *negative*, then for any $\mathcal{L}$-variable x,

(EI) $\vdash_L Ex \leftrightarrow x = x$;

Proof By (FUI*), $\vdash_L \forall x(x = x) \supset (Ex \supset x = x)$. By (∀=R), $\vdash_L \forall x(x = x)$. So $\vdash_L Ex \supset x = x$. Conversely, $x = x \supset Ex$ by (Neg). ■

Lemma 4.10 (Symmetry and transitivity of identity)
For any $\mathcal{L}$-variables x, y, z,

(=S) $\vdash_L x = y \supset y = x$;

(=T) $\vdash_L x = y \supset y = z \supset x = z$.

PROOF For $(=\mathrm{S})$, let v be some variable $\notin \{x, y\}$. Then

1. $\vdash_L v{=}y \supset (v{=}x \supset y{=}x)$. (LL*)
2. $\vdash_L x{=}y \supset (x{=}x \supset y{=}x)$. (1, (Sub*))
3. $\vdash_L x{=}y \supset x{=}x$. ((=R), or (Neg) and (∀=R))
4. $\vdash_L x{=}y \supset y{=}x$. (2, 3)

For $(=\mathrm{T})$,

1. $\vdash_L x{=}y \supset y{=}x$. (=S)
2. $\vdash_L y{=}x \supset (y{=}z \supset x{=}z)$. (LL*)
3. $\vdash_L x{=}y \supset (y{=}z \supset x{=}z)$. (1, 2)

■

Next we have proof-theoretic analogues of lemmas 3.8 and 3.13:

LEMMA 4.11 (SYNTACTIC ALPHA-CONVERSION)
If A, A' are $\mathcal{L}$-formulas, and A' is an alphabetic variant of A, then

(AC) $\vdash_L A \leftrightarrow A'$.

PROOF by induction on A.

1. A is atomic. Then $A = A'$ and $A \leftrightarrow A'$ is a propositional tautology.
2. A is $\neg B$. Then A' is $\neg B'$, where B' is an alphabetic variant of A'. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (PC), $\vdash_L \neg B \leftrightarrow \neg B'$.
3. A is $B \supset C$. Then A' is $B' \supset C'$, where B', C' are alphabetic variants of B, C, respectively. By induction hypothesis, $\vdash_L B \leftrightarrow B'$ and $\vdash_L C \leftrightarrow C'$. So by (PC), $\vdash_L (B \supset C) \leftrightarrow (B' \supset C')$.
4. A is $\forall xB$. Then A' is either $\forall xB'$ or $\forall z[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that A' is $\forall xB'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (UG) and (UD), $\vdash_L \forall xB \leftrightarrow \forall xB'$.

   Alternatively, assume B is $\forall z[z/x]B'$ and $z \notin \mathit{Var}(B')$. Since B' differs from B at most in renaming bound variables, if z were free in B, then $z \in \mathit{Var}(B')$. So z is not free in

B. Then

1. $\vdash_L B \leftrightarrow B'$ (induction hypothesis)
2. $\vdash_L [z/x]B \leftrightarrow [z/x]B'$ (1, (Sub*))
3. $\vdash_L \forall xB \supset Ez \supset [z/x]B$ (FUI*)
4. $\vdash_L \forall xB \supset Ez \supset [z/x]B'$ (2, 3)
5. $\vdash_L \forall z\forall xB \supset \forall zEz \supset \forall z[z/x]B'$ (4, (UG), (UD))
6. $\vdash_L \forall zEz$ ($\forall$Ex)
7. $\vdash_L \forall z\forall xB \supset \forall z[z/x]B'$ (5, 6)
8. $\vdash_L \forall xB \supset \forall z\forall xB$ ((VQ), z not free in B)
9. $\vdash_L \forall xB \supset \forall z[z/x]B'$. (7, 8)

Conversely,

10. $\vdash_L \forall z[z/x]B' \supset Ex \supset [x/z][z/x]B'$ (FUI*)
11. $\vdash_L \forall z[z/x]B' \supset Ex \supset B$ (1, 10, $z \notin \mathit{Var}(B')$)
12. $\vdash_L \forall x\forall z[z/x]B' \supset \forall xB$ (11, (UG), (UD), ($\forall$Ex))
13. $\vdash_L \forall z[z/x]B' \supset \forall x\forall z[z/x]B'$ (VQ)
14. $\vdash_L \forall z[z/x]B' \supset \forall xB$ (12, 13)

5. A is $\Box B$. Then A' is $\Box B'$, where B' is an alphabetic variant of B. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. Then by (Nec), $\vdash_L \Box(B \leftrightarrow B')$, and by (K) and (PC), $\vdash_L \Box B \leftrightarrow \Box B'$. ■

Lemma 4.12 (Closure under transformations)
For any $\mathcal{L}$-formula A and transformation τ on $\mathcal{L}$,

(Sub$^\tau$) $\vdash_L A$ iff $\vdash_L A^\tau$.

Proof Assume $\vdash_L A$. Let $x_1, \dots, x_n$ be the variables in A. If $n = 0$, then $A = A^\tau$ and the result is trivial. If $n = 1$, then A^τ is $[x_1^\tau/x_1]A$, and x_1^τ is either x_1 itself or does not occur in A. In the first case, $[x_1^\tau/x_1]A = A$ and the result is again trivial. In the second case, x_1^τ is modally free for x_1 in A, and thus $\vdash_L [x_1^\tau/x_1]A$ by (Sub*).

Assume then that $n > 1$. Note first that $A^\tau = [x_n^\tau/v_n]\dots[x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2]\dots[v_n/x_n]A$, where $v_2, \dots, v_n$ are distinct variables not in A or A^τ. This is easily shown by induction on the subformulas B of A (ordered by complexity). To keep things short, let Σ abbreviate $[x_n^\tau/v_n]\dots[x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2]\dots[v_n/x_n]$.

1. If B is $Px_j \dots x_k$, then $x_j, \dots, x_k$ are variables from $x_1, \dots, x_n$, and $\Sigma B = Px_j^\tau \dots x_k^\tau = B^\tau$, by definitions 3.3 and 3.11.
2. If B is $\neg C$, then by induction hypothesis, $\Sigma C = C^\tau$, and hence $\neg\Sigma C = \neg C^\tau$. But $\Sigma\neg C$ is $\neg\Sigma C$ by definition 3.3, and $(\neg C)^\tau$ is $\neg C^\tau$ by definition 3.11.

3. The case for $C \supset D$ is analogous.

4. If B is $\forall zC$, then by induction hypothesis, $\Sigma C = C^\tau$. Since τ is injective, $\Sigma\forall zC$ is $\forall\Sigma z\Sigma C$ by definition 3.3, and $(\forall zC)^\tau$ is $\forall z^\tau C^\tau$ by definition 3.11. Moreover, since z is one of $x_1, \ldots, x_n$, $\Sigma z = z^\tau$.

5. If B is $\Box C$, then by induction hypothesis, ΣC is C^τ, and hence $\Box\Sigma C$ is $\Box C^\tau$. But $\Sigma\Box C$ is $\Box\Sigma C$ by definition 3.3, and $(\Box C)^\tau$ is $\Box C^\tau$ by definition 3.11.

Now we show that L contains all "segments" of $[x_n^\tau/v_n]\ldots[x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$, beginning with the rightmost substitution, $[v_n/x_n]A$. Since v_n is modally free for x_n in A, by (Sub*), $\vdash_L [v_n/x_n]A$. Likewise, for each $1 < i < n$, v_i is modally free for x_i in $[v_{i+1}/x_{i+1}]\ldots[v_n/x_n]A$. So $\vdash_L [v_2/x_2]\ldots[v_n/x_n]A$.

With respect to $[x_1^\tau/x_1]$, we distinguish three cases. First, if $x_1 = x_1^\tau$, then $\vdash_L [x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$, because $[x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$ is $[v_2/x_2]\ldots[v_n/x_n]A$. Second, if $x_1 \neq x_1^\tau$ and $x_1^\tau \notin \mathit{Var}(A)$, then $x_1^\tau \notin \mathit{Var}([v_2/x_2]\ldots[v_n/x_n]A)$, since the $v_1, \ldots, v_n$ are not in $\mathit{Var}(A)$ or $\mathit{Var}(A^\tau)$. So x_1^τ is modally free for x_1 in $[v_2/x_2]\ldots[v_n/x_n]A$, and by (Sub*), $\vdash_L [x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$. Third, if $x_1 \neq x_1^\tau$ and $x_1^\tau \in \mathit{Var}(A)$, then x_1^τ must be one of $x_2, \ldots, x_n$. Then again $x_1^\tau \notin \mathit{Var}([v_2/x_2]\ldots[v_n/x_n]A)$, and so $\vdash_L [x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$ by (Sub*).

Next, x_2^τ is modally free for v_2 in $[x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$, because τ is injective and hence $x_2^\tau \neq x_1^\tau$, so x_2^τ does not occur in $[x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$. Hence $\vdash_L [x_2^\tau/v_2][x_1^\tau/x_1]$ $[v_2/x_2]\ldots[v_n/x_n]A$. By the same reasoning, for each $2 < i \leq n$, x_i^τ is modally free for v_i in $[x_{i-1}^\tau/v_{i-1}]\ldots[x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2]\ldots[v_n/x_n]A$. So $\vdash_L [x_n^\tau/v_n]\ldots[x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2]\ldots$ $[v_n/x_n]A$, i.e. $\vdash_L A^\tau$.

This proves the left-to-right direction of (Sub$^\tau$). The other direction immediately follows. Let $x_1^\tau, \ldots, x_n^\tau$ be the variables in A^τ, and let σ be an arbitrary transformation that maps each x_i^τ back to x_i (i.e., to $(x_i^\tau)^{\tau^{-1}}$). By the left-to-right direction of (Sub$^\tau$), $\vdash_L A^\tau$ entails $\vdash_L (A^\tau)^\sigma$, and $(A^\tau)^\sigma$ is simply A. ■

Lemma 4.13 (Leibniz' Law with partial substitution)
Let A be a formula of $\mathcal{L}$, and x, y variables of $\mathcal{L}$. Let $[y//x]A$ be A with *one or more* occurrences of x replaced by y.

(LL$_\text{p}^*$) $\vdash_L x{=}y \supset A \supset [y//x]A$, provided the following conditions are satisfied.

(i) $[y//x]A$ does not replace any occurrence of x in the scope of a quantifier binding x or y.

(ii) Either y is modally free for x in A, or $[y//x]A$ does not replace any occurrence of x in the scope of a modal operator in A that also contains y.

(iii) In the scope of any modal operator in A, $[y//x]A$ either replaces all or no occurrences of x by y.

Proof Let $v \neq y$ be a variable not in $Var(A)$, and let $[v//x]A$ be like $[y//x]A$ except that all new occurrences of y are replaced by v: if $[y//x]A$ satisfies (i)–(iii), then so does $[y//x]A$ with all new occurrences of y replaced by v. Moreover, in the resulting formula $[v//x]A$ all occurrences of v are free and free for y, by clause (i); so $[y/v][v//x]A = [y//x]A$ by definition 3.3. By (LL*),

(1) $\vdash_L v = y \supset [v//x]A \supset [y/v][v//x]A,$

provided that y is modally free for v in $[v//x]A$, i.e. provided that either y is modally free for x in A, or $[v//x]A$ (and thus $[y//x]A$) does not replace any occurrence of x in the scope of a modal operator in A that also contains y. This is guaranteed by condition (ii). Since $[y/v][v//x]A$ is $[y//x]A$, (1) can be shortened to

(2) $\vdash_L v = y \supset [v//x]A \supset [y//x]A.$

By (Sub*), it follows that

(3) $\vdash_L [x/v](v = y \supset [v//x]A \supset [y//x]A),$

provided that x is modally free for v in $v = y \supset [v//x]A \supset [y//x]A$. Since this isn't a formula of the form $\Box B$, x is modally free for v here iff no free occurrences of x and v lie in the scope of the same modal operator in $[v//x]A$. So whenever $[v//x]A$ (and thus $[y//x]A$) replaces some occurrences of x in the scope of a modal operator in A, then it must replace all occurrences of x in the scope of that operator. This is guaranteed by condition (iii). By definition 3.3, (3) can be simplified to

(4) $\vdash_L x = y \supset A \supset [y//x]A.$

■

I will never actually use (LL^*_p). I mention it only because Leibniz' Law is often stated for partial substitutions, and you may have wondered how the corresponding version looks in the present systems. Now you know. We could indeed have used (LL^*_p) as basic axiom instead of (LL*); (LL*) would then be derivable, because every formula A has an alphabetic variant A' such that $[y/x]A$ is an instance of $[y//x]A'$ that satisfies (i)–(iii) iff y is modally free for x in A, and because (LL*) is not used in the proof of lemma 4.11.

I have chosen (LL*) as basic due to its much greater simplicity.[5]

LEMMA 4.14 (LEIBNIZ' LAW WITH SEQUENCES)
For any $\mathcal{L}$-formula A and variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(LL$^*_\mathrm{n}$) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$, provided each y_i is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{n-1}]A$.

For $i = 1$, the proviso is meant to say that y_1 is modally free for x_1 in A.

PROOF By induction on n. For $n = 1$, (LL$^*_\mathrm{n}$) is (LL*). Assume then that $n > 1$ and that each y_i in $y_1, \ldots, y_n$ is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{n-1}]A$. Let z be some variable not in A, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$. So z is modally free for x_n in A. By (LL*),

(1) $\vdash_L x_n = z \supset A \supset [z/x_n]A$.

By induction hypothesis,

(2) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset [z/x_n]A \supset [y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$.

By assumption, y_n is modally free for x_n in $[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}]A$. Then y_n is also modally free for z in $[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$. So by (LL*),

(3) $\vdash_L z = y_n \supset [y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A \supset$
$[y_n/z][y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$.

But $[y_n/z][y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$ is $[y_1, \ldots, y_n/x_1, \ldots, x_n]A$. Combining (1)–(3), we therefore have

(4) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset x_n = z \wedge z = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$.

---

5 Kutz's system uses the following version of (LL$^*_\mathrm{p}$) ([Kutz 2000: 43]):

(LL$^\mathrm{K}_\mathrm{p}$) $\vdash x = y \supset A \supset [y//x]A$, provided that
(i) x is free in A and y is free for x in A,
(ii) y is not free in the scope of a modal operator in A, and
(iii) in the scope of any modal operator in A, $[y//x]A$ either replaces all or no occurrences of x by y.

Evidently, this is a lot more restrictive than (LL$^*_\mathrm{p}$). For example, (LL$^*_\mathrm{p}$) validates

$\vdash x = y \supset \Box Gxy \supset \Box Gyy$ and
$\vdash x = y \supset (Fx \vee \Diamond Gxy) \supset (Fy \vee \Diamond Gxy)$,

which can't be derived in Kutz's system.

So by (Sub*),

(5) $\quad \vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset x_n = x_n \wedge x_n = y_n \supset A \supset [y_1, \ldots, y_n / x_1, \ldots, x_n]A.$

Since $\vdash_L x_n = y_n \supset x_n = x_n$ (by either (=R) or (Neg) and ($\forall$=R)), it follows that

(6) $\quad \vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset [y_1, \ldots, y_n / x_1, \ldots, x_n]A.$

■

LEMMA 4.15 (CROSS-SUBSTITUTION)
For any $\mathcal{L}$-formula A and variables x, y,

(CS) $\vdash_L x = y \supset \Box A \supset \Box(y = z \supset [z/x]A)$, provided z is not free in A.

More generally, for any variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(CS$_\mathrm{n}$) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset \Box A \supset \Box(y_1 = z_1 \wedge \ldots \wedge y_n = z_n \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A)$, provided none of $z_1, \ldots, z_n$ is free in A.

PROOF For (CS), assume z is not free in A. Then

1. $\vdash_L x = z \supset A \supset [z/x]A.$ (LL*)
2. $\vdash_L A \supset (x = z \supset [z/x]A).$ (1)
3. $\vdash_L \Box A \supset \Box(x = z \supset [z/x]A).$ (2, (Nec), (K))
4. $\vdash_L x = y \supset \Box(x = z \supset [z/x]A) \supset \Box(y = z \supset [z/x]A).$ (LL*)
5. $\vdash_L x = y \supset \Box A \supset \Box(y = z \supset [z/x]A).$ (3, 4)

Step 4 is justified by the fact that x is not free in $[z/x]A$ and so x and y are modally separated in $x = z \supset [z/x]A$.

The proof for (CS$_\mathrm{n}$) is analogous. Assume none of $z_1, \ldots, z_n$ is free in A. Then

1. $\vdash_L x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset A \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A.$ (LL$^*_\mathrm{n}$)
2. $\vdash_L A \supset (x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A).$ (1)
3. $\vdash_L \Box A \supset \Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A).$ (2, (Nec), (K))
4. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset$
   $\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A) \supset$
   $\Box(y_1 = z_1 \wedge \ldots \wedge y_n = z_n \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A).$ (LL$^*_\mathrm{n}$)
5. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset \Box A \supset$
   $\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n / x_1, \ldots, x_n]A).$ (3, 4)

Step 4 is justified by the fact that none of $x_1, \ldots, x_n$ is free in $[z_1, \ldots, z_n/x_1, \ldots, x_n]A$, and each y_i is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{i-1}]\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, i.e. in $\Box(y_1 = z_1 \wedge \ldots \wedge y_{i-1} = z_{i-1} \wedge x_i = z_i \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, because x_i and y_i are modally separated in $y_1 = z_1 \wedge \ldots \wedge y_{i-1} = z_{i-1} \wedge x_i = z_i \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A$. ■

Lemma 4.16 (Substitution-free Universal Instantiation)
For any $\mathcal{L}$-formula A and variables x, y,

(FUI**) $\vdash_L \forall xA \supset (Ey \supset \exists x(x = y \wedge A))$.

Proof Let z be a variable not in $Var(A), x, y$.

1.	$\vdash_L z = y \supset Ey \supset Ez$	(LL*)
2.	$\vdash_L \forall xA \supset Ez \supset [z/x]A$	((FUI*), $z \notin Var(A)$)
3.	$\vdash_L \forall xA \wedge Ey \supset z = y \supset [z/x]A$	(1, 2)
4.	$\vdash_L \forall x(x = z \supset \neg A) \supset Ez \supset (z = z \supset [z/x]\neg A)$	((FUI*), $z \notin Var(A)$)
5.	$\vdash_L Ez \supset z = z$	((=R), or (∀=R), (FUI*))
6.	$\vdash_L \forall x(x = z \supset \neg A) \supset Ez \supset [z/x]\neg A$	(4, 5)
7.	$\vdash_L Ez \supset [z/x]A \supset \exists x(x = z \wedge A)$	(6)
8.	$\vdash_L \forall xA \wedge Ey \supset z = y \supset \exists x(x = z \wedge A)$	(1, 3, 7)
9.	$\vdash_L z = y \supset \exists x(x = z \wedge A) \supset \exists x(x = y \wedge A)$	((LL*), $z \notin Var(A)$)
10.	$\vdash_L \forall xA \wedge Ey \supset z = y \supset \exists x(x = y \wedge A)$	(8, 9)
11.	$\vdash_L \forall z(\forall xA \wedge Ey) \supset \forall z(z = y \supset \exists x(x = y \wedge A))$	(10, (UG), (UD))
12.	$\vdash_L \forall xA \wedge Ey \supset \forall z(z = y \supset \exists x(x = y \wedge A))$	(11, (VQ))
13.	$\vdash_L \forall z(z = y \supset \exists x(x = y \wedge A)) \supset y = y \supset \exists x(x = y \wedge A)$	((FUI*), $z \notin Var(A)$)
14.	$\vdash_L Ey \supset y = y$	((=R), or (∀=R), (FUI*))
15.	$\vdash_L \forall z(z = y \supset \exists x(x = y \wedge A)) \supset Ey \supset \exists x(x = y \wedge A)$	(13, 14)
16.	$\vdash_L \forall xA \supset Ey \supset \exists x(x = y \wedge A)$	(12, 15)

■

Incidentally, (FUI*) can also be derived from (FUI**), so we could just as well have used (FUI**) as basic axiom instead of (FUI*).

5 Logics with explicit substitution

Let's move on to languages with substitution. We first have to lay down some axioms governing the substitution operator. An obvious suggestion would be the lambda-conversion principle

$$\langle y : x\rangle A \leftrightarrow [y/x]A,$$

which would allow us to move back and forth between e.g. $\langle y : x\rangle Fx$ and Fy. But we've seen in lemma 3.9 that if things can have multiple counterparts, then these transitions are sound only under certain conditions: the move from $\langle y : x\rangle A$ to $[y/x]A$ requires that y is modally free for x in A, the other direction requires that y and x are modally separated in A. So we have the following somewhat more complex principles:

(SC1) $\langle y : x\rangle A \leftrightarrow [y/x]A$, provided y and x are modally separated in A.

(SC2) $\langle y : x\rangle A \supset [y/x]A$, provided y is modally free for x in A.

But now we need further principles telling us how $\langle y : x\rangle$ behaves when y is not modally free for x. For example, $\langle y : x\rangle\neg A$ should always entail $\neg\langle y : x\rangle A$, even if y is not modally free for x in A. More generally, the substitution operator commutes with every non-modal operator as long as there is no clash of bound variables:

(S$\neg$) $\langle y : x\rangle\neg A \leftrightarrow \neg\langle y : x\rangle A$,

(S$\supset$) $\langle y : x\rangle(A \supset B) \leftrightarrow (\langle y : x\rangle A \supset \langle y : x\rangle B)$,

(S$\forall$) $\langle y : x\rangle\forall zA \leftrightarrow \forall z\langle y : x\rangle A$, provided $z \notin \{x, y\}$,

(SS1) $\langle y : x\rangle\langle y_2 : z\rangle A \leftrightarrow \langle y_2 : z\rangle\langle y : x\rangle A$, provided $z \notin \{x, y\}$ and $y_2 \neq x$.

Substitution does not commute with the box. Roughly speaking, this is because $\langle y : x\rangle\Box A(x, y)$ says that at all accessible worlds, all counterparts x' and y' of y are $A(x', y')$, while $\Box\langle y : x\rangle A(x, y)$ says that at all accessible worlds, every counterpart $x' = y'$ of y is such that $A(x', y')$. In the first case, x' and y' may be different counterparts of y, while in the second case, they must be the same. Thus $\langle y : x\rangle\Box A$ entails $\Box\langle y : x\rangle A$, but the other direction holds only if either y does not have multiple counterparts at accessible worlds (relative to the same counterpart relation), or at most one of x and y occurs freely in A (including the special case where x and y are the same variable).

(S$\Box$) $\langle y : x\rangle\Box A \supset \Box\langle y : x\rangle A$,

(S$\Diamond$) $\langle y : x\rangle\Diamond A \supset \Diamond\langle y : x\rangle A$, provided at most one of x, y is free in A.

These principles largely make (SC1) and (SC2) redundant. We only need to add the special case for substituting free variables in atomic formulas and in substitution operators, as well as a principle for vacuous substitutions:

(SAt) $\langle y : x\rangle Px_1 \ldots x_n \leftrightarrow P[y/x]x_1 \ldots [y/x]x_n$.

(SS2) $\langle y : x\rangle\langle x : z\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$.

(VS) $A \leftrightarrow \langle y : x \rangle A$, provided x is not free in A.

Lemma 5.1 (Soundness of the substitution axioms)
If $\mathcal{L}_s$ is a language of quantified modal logic with substitution, then every $\mathcal{L}_s$-instance of (S¬), (S⊃), (S∀), (SS1), (S□), (S◇), (SAt), (SS2), and (VS) is valid in every (positive or negative) counterpart model.

Proof

1. (S¬). $w, V \Vdash \langle y : x \rangle \neg A$ iff $w, V^{[y/x]} \Vdash \neg A$ by definition 3.2, iff $w, V^{[y/x]} \nVdash A$ by definition 2.7, iff $w, V \nVdash \langle y : x \rangle A$ by definition 3.2, iff $w, V \Vdash \neg \langle y : x \rangle A$ by definition 2.7.

2. (S⊃). $w, V \Vdash \langle y : x \rangle (A \supset B)$ iff $w, V^{[y/x]} \Vdash A \supset B$ by definition 3.2, iff $w, V^{[y/x]} \nVdash A$ or $w, V^{[y/x]} \Vdash B$ by definition 2.7, iff $w, V \nVdash \langle y : x \rangle A$ or $w, V \Vdash \langle y : x \rangle B$ by definition 3.2, iff $w, V \Vdash \langle y : x \rangle A \supset \langle y : x \rangle B$ by definition 2.7.

3. (S∀). Assume $z \notin \{x, y\}$. Then the existential z-variants V' of $V^{[y/x]}$ on w coincide at w with the functions $(V^*)^{[y/x]}$ where V^* is an existential z-variant V^* of V on w. And so $w, V \Vdash \langle y : x \rangle \forall z A$ iff $w, V^{[y/x]} \Vdash \forall z A$ by definition 3.2, iff $w, V' \Vdash A$ for all existential z-variants V' of $V^{[y/x]}$ on w by definition 2.7, iff $w, (V^*)^{[y/x]} \Vdash A$ for all existential z-variants V^* of V on w, iff $w, V^* \Vdash \langle y : x \rangle A$ for all existential z-variants V^* of V on w by definition 3.2, iff $w, V \Vdash \forall z \langle y : x \rangle A$ by definition 2.7.

4. (SS1). Assume $z \notin \{x, y\}$ and $y_2 \neq x$. Then the function $[y/x] \cdot [y_2/z]$ is identical to the function $[y_2/z] \cdot [y/x]$. So $w, V \Vdash \langle y : x \rangle \langle y_2 : z \rangle A$ iff $w, V^{[y/x] \cdot [y_2/z]} \Vdash A$ by definition 3.2, iff $w, V^{[y_2/z] \cdot [y/x]} \Vdash A$, iff $w, V \Vdash \langle y_2 : z \rangle \langle y : x \rangle A$ by definition 3.2.

5. (S□). Assume $w, V \nVdash \Box \langle y : x \rangle A$. By definitions 2.7 and 3.2, this means that $w', V'^{[y/x]} \nVdash A$ for some w', V' such that wRw' and $V_w \triangleright V'_{w'}$, i.e. there is a $C \in K_{w,w'}$ for which $V'_{w'}$ assigns to every variable z a C-counterpart of its value under V_w (or nothing if there is none). Then for all z, $V'^{[y/x]}_{w'}(z)$ is a C-counterpart of $V^{[y/x]}_w(z)$ (or undefined if there is none), since $V'^{[y/x]}_{w'}(x) = V'_{w'}(y)$ is a C-counterpart of $V_w(y) = V^{[y/x]}_w(x)$ (or undefined if there is none). So $V^{[y/x]}_w \triangleright V'^{[y/x]}_{w'}$. And so $w', V^* \nVdash A$ for some w', V^* such that wRw' and $V^{[y/x]}_w \triangleright V^*_{w'}$. So $w, V \Vdash \langle y : x \rangle \Box A$ by definitions 2.7 and 3.2.

6. (S◇). Assume $w, V \Vdash \langle y : x \rangle \Diamond A$ and at most one of x, y is free in A. By definitions 2.7 and 3.2, $w', V^* \Vdash A$ for some w', V^* such that wRw' and $V^{[y/x]}_w \triangleright V^*_{w'}$, i.e. there is a $C \in K_{w,w'}$ for which $V^*_{w'}$ assigns to every variable z a C-counterpart of its value under V_w (or nothing if there is none). We have to show that there is a w'-image V' of V at w such that $w, V'^{[y/x]} \Vdash A$, since then $w, V \Vdash \Diamond \langle y : x \rangle A$.

   If x is the same variable as y, then $V^*_{w'}(x) = V^*_{w'}(y)$ is a C-counterpart at w' of $V^{[y/x]}_w(x) = V^{[y/x]}_w(y) = V_w(x) = V_w(y)$ at w (or undefined if there is none), so we can choose V^* itself as V'. We then have $w, V'^{[y/x]} \Vdash A$ because $V'^{[y/x]} = V'$.

Else if x is not free in A, let V' be some x-variant of V^* at w' such that $V^*_{w'}(x)$ is some C-counterpart at w' of $V_w(x)$ at w (or undefined if there is none). Since $V^*_{w'}(y)$ is a C-counterpart at w' of $V^{[y/x]}_w(y) = V_w(y)$ at w (or undefined if there is none), V' is a w'-image of V at w. Moreover, $V'^{[y/x]}$ and V^* agree at w' about all variables other than x; so by the coincidence lemma 2.9, $w', V'^{[y/x]} \Vdash A$.

Else if y is not free in A, let V' be like V^* except that $V'_{w'}(y) = V^*_{w'}(x)$ and $V'_{w'}(x)$ is some C-counterpart at w' of $V_w(x)$ at w (or undefined if there is none). Since $V'_{w'}(y) = V^*_{w'}(x)$ is a C-counterpart at w' of $V^{[y/x]}_w(x) = V_w(y)$ at w (or undefined if there is none), V' is a w'-image of V at w. Moreover, $V'^{[y/x]}$ and V^* agree at w' about all variables other than y; in particular, $V'^{[y/x]}_{w'}(x) = V'_{w'}(y) = V^*_{w'}(x)$. So by the coincidence lemma 2.9, $w', V'^{[y/x]} \Vdash A$.

7. (SAt). $w, V \Vdash \langle y : x\rangle Px_1 \dots x_n$ iff $w, V^{[y/x]} \Vdash Px_1 \dots x_n$ by definition 3.2, iff $w, V \Vdash [y/x]Px_1 \dots x_n$ by lemma 3.9.
8. (SS2). $w, V \Vdash \langle y : x\rangle\langle x : z\rangle A$ iff $w, V^{[y/x]\cdot[x/z]} \Vdash A$ by definition 3.2, iff $w, V^{[y/z]\cdot[y/x]} \Vdash A$ because $[y/x] \cdot [x/z] = [y/z] \cdot [y/x]$, iff $w, V \Vdash \langle y : z\rangle\langle y : x\rangle A$ by definition 3.2.
9. (VS). By definition 3.2, $w, V \Vdash \langle y : x\rangle A$ iff $w, V^{[y/x]} \Vdash A$. If x is not free in A, then $V^{[y/x]}$ agrees with V at w about all free variables in A. So by the coincidence lemma 2.9, $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash A$. So then $w, V \Vdash \langle y : x\rangle A$ iff $w, V \Vdash A$. ■

Definition 5.2 (Positive logics with substitution)
Given a language $\mathcal{L}_s$ with substitution, a *positive (quantified modal) logic with substitution* in $\mathcal{L}_s$ is a set of formulas $L \subseteq \mathcal{L}_s$ that contains all $\mathcal{L}_s$-instances of the substitution axioms (S$\neg$), (S$\supset$), (S$\forall$), (SS1), (S$\Box$), (S$\Diamond$), (SAt), (SS2), (VS), as well as (Taut), (UD), (VQ), ($\forall$Ex), (=R), (K),

(FUI$_\mathrm{s}$) $\forall x A \supset (Ey \supset \langle y : x\rangle A)$,

(LL$_\mathrm{s}$) $x = y \supset (A \supset \langle y : x\rangle A)$,

and that is closed under (MP), (UG), (Nec) and

(Sub$_\mathrm{s}$) if $\vdash_L A$, then $\vdash_L \langle y : x\rangle A$.

The smallest such logic is called P_s.

Definition 5.3 (Negative logics with substitution)
Given a language $\mathcal{L}_s$ with substitution, a *negative (quantified modal) logic with substitution* in $\mathcal{L}_s$ is a set $L \subseteq \mathcal{L}_s$ that contains all $\mathcal{L}_s$-instances of the substitution axioms (S$\neg$), (S$\supset$), (S$\forall$), (SS1), (S$\Box$), (S$\Diamond$), (SAt), (SS2), (VS), as well as (Taut), (UD), (VQ), (Neg), (NA), ($\forall$=R), (K), (FUI$_\mathrm{s}$), (LL$_\mathrm{s}$), and that is closed under (MP), (UG), (Nec) and (Sub$_\mathrm{s}$). The smallest such logic is called N_s.

Theorem 5.4 (Soundness of P_s)
Every member of P_s is valid in every positive counterpart model.

Proof We have to show that all P_s axioms are valid in every model, and that validity is closed under (MP), (UG), (Nec) and (Sub$_\mathrm{s}$). For (Taut), (UD), (VQ), (∀Ex), (=R), (K), (MP), (UG), (Nec), see the proof of theorem 4.3. For the substitution axioms, see lemma 5.1. The remaining cases are (FUI$_\mathrm{s}$), (LL$_\mathrm{s}$), and (Sub$_\mathrm{s}$).

1. (FUI$_\mathrm{s}$). Assume $w, V \Vdash \forall x A$ and $w, V \Vdash Ey$ in some model. By definition 2.7, the latter means that $V_w(y) \in D_w$, and the former means that $w, V' \Vdash A$ for all existential x-variants V' of V on w. So in particular, $w, V' \Vdash A$, where V' is the x-variant of V on w with $V_w(x) = V_w(y)$. So $w, V \Vdash \langle y : x \rangle A$ by definition 3.2.
2. (LL$_\mathrm{s}$). Assume $w, V \Vdash x = y$ and $w, V \Vdash A$. By definitions 2.7 and 2.3, then $V_w(x) = V_w(y)$. So $w, V \Vdash \langle y : x \rangle A$ by definition 3.2.
3. (Sub$_\mathrm{s}$). Assume $w, V \nVdash \langle y : x \rangle A$ in some model $\mathcal{M} = \langle \mathcal{S}, V \rangle$. By definition 3.2, then $w, V' \nVdash A$, where V' is the x-variant of V on w with $V'(x) = V(y)$. So A is invalid in the model $\langle \mathcal{S}, V' \rangle$. Hence if A is valid in all positive models, then so is $\langle y : x \rangle A$. ■

Theorem 5.5 (Soundness of N_s)
Every member of N_s is valid in every negative counterpart model.

Proof All the cases needed here are covered in the proofs of theorem 4.6 and 5.4. ■

To derive some further properties of these systems, let $\mathcal{L}$ range over languages of quantified modal logic with substitution, and L over positive or negative logics in $\mathcal{L}$.

Closure under propositional consequence and the validity of (∀Ex) and (∀=R) are proved just as for substitution-free logics (see lemmas 4.7 and 4.8). So we move on immediately to more interesting properties.

Lemma 5.6 (Substitution expansion)
If A is an $\mathcal{L}$-formula and x, y, z $\mathcal{L}$-variables, then

(SE1) $\vdash_L A \leftrightarrow \langle x : x \rangle A$;

(SE2) $\vdash_L \langle y : x \rangle A \leftrightarrow \langle y : z \rangle \langle z : x \rangle A$, provided z is not free in A.

Proof (SE1) is proved by induction on A.

1. A is atomic. Then $\vdash_L \langle x:x\rangle A \leftrightarrow [x/x]A$ by (SAt), and so $\vdash_L \langle x:x\rangle A \leftrightarrow A$ because $[x/x]A = A$.

2. A is $\neg B$. By induction hypothesis, $\vdash_L B \leftrightarrow \langle x:x\rangle B$. So by (PC), $\vdash_L \neg B \leftrightarrow \neg\langle x:x\rangle B$. And by $\langle S\neg\rangle$, $\vdash_L \langle x:x\rangle\neg B \leftrightarrow \neg\langle x:x\rangle B$.

3. A is $B \supset C$. By induction hypothesis, $\vdash_L B \leftrightarrow \langle x:x\rangle B$ and $\vdash_L C \leftrightarrow \langle x:x\rangle C$. So $\vdash_L (B \supset C) \leftrightarrow (\langle x:x\rangle B \supset \langle x:x\rangle C)$. And by $\langle S\supset\rangle$, $\vdash_L \langle x:x\rangle(B \supset C) \leftrightarrow (\langle x:x\rangle B \supset \langle x:x\rangle C)$.

4. A is $\forall zB$. If $z = x$, then $\vdash_L \forall xB \leftrightarrow \langle x:x\rangle\forall xB$ by (VS). If $z \neq x$, then by induction hypothesis, $\vdash_L B \leftrightarrow \langle x:x\rangle B$; by (UG) and (UD), $\vdash_L \forall zB \leftrightarrow \forall z\langle x:x\rangle B$; and $\vdash_L \langle x:x\rangle\forall zB \leftrightarrow \forall z\langle x:x\rangle B$ by (S$\forall$).

5. A is $\langle y:z\rangle B$. If $z = x$, then $\vdash_L \langle y:x\rangle B \leftrightarrow \langle x:x\rangle\langle y:x\rangle B$ by (VS). If $z \neq x$, then by induction hypothesis, $\vdash_L B \leftrightarrow \langle x:x\rangle B$; by (Sub$_s$) and (S$\supset$), $\vdash_L \langle y:z\rangle B \leftrightarrow \langle y:z\rangle\langle x:x\rangle B$; and $\vdash_L \langle x:x\rangle\langle y:z\rangle B \leftrightarrow \langle y:z\rangle\langle x:x\rangle B$ by (SS1) (if $y \neq x$) or (SS2) (if $y = x$).

6. A is $\Box B$. By (S$\Box$), $\vdash_L \langle x:x\rangle\Box B \supset \Box\langle x:x\rangle B$. Conversely, since at most one of x, x is free in $\neg B$, by (S$\Diamond$), $\vdash_L \langle x:x\rangle\Diamond\neg B \supset \Diamond\langle x:x\rangle\neg B$. Contraposing and unraveling the definition of the diamond, we have $\vdash_L \Box\neg\langle x:x\rangle\neg B \supset \neg\langle x:x\rangle\neg\Box\neg\neg B$. Since $\vdash_L \Box\neg\langle x:x\rangle\neg B \leftrightarrow \Box\langle x:x\rangle B$ and $\vdash_L \neg\langle x:x\rangle\neg\Box\neg\neg B \leftrightarrow \langle x:x\rangle B$ (by (S$\neg$), (Sub$_s$), (S$\supset$), (Nec) and (K)), this means that $\vdash_L \Box\langle x:x\rangle B \supset \langle x:x\rangle\Box B$.

As for (SE2): by (VQ), $\vdash_L \langle y:x\rangle A \leftrightarrow \langle y:z\rangle\langle y:x\rangle A$. And $\vdash_L \langle y:x\rangle\langle y:z\rangle A \leftrightarrow \langle y:z\rangle\langle y:x\rangle A$ by (SS1) (if $y \neq x$) or (SS2) (if $y = x$). Moreover, by (SS2), $\vdash_L \langle y:z\rangle\langle z:x\rangle A \leftrightarrow \langle y:x\rangle\langle y:z\rangle A$. So by (PC), $\vdash_L \langle y:x\rangle A \leftrightarrow \langle y:z\rangle\langle z:x\rangle A$. ■

Lemma 5.7 (Substituting bound variables)
For any $\mathcal{L}$-sentence A and variables x, y,

(SBV) $\vdash_L \forall xA \leftrightarrow \forall y\langle y:x\rangle A$, provided y is not free in A.

Proof

1. $\vdash_L \forall y\langle y:x\rangle A \supset Ex \supset \langle x:y\rangle\langle y:x\rangle A.$ (FUI$_s$)
2. $\vdash_L \langle x:y\rangle\langle y:x\rangle A \leftrightarrow A.$ ((SE1), (SE2))
3. $\vdash_L \forall x\forall y\langle y:x\rangle A \supset \forall xEx \supset \forall xA.$ (1, 2, (UG), (UD))
4. $\vdash_L \forall x\forall y\langle y:x\rangle A \supset \forall xA.$ (3, ($\forall$Ex))
5. $\vdash_L \forall y\langle y:x\rangle A \supset \forall x\forall y\langle y:x\rangle A.$ (VQ)
6. $\vdash_L \forall y\langle y:x\rangle A \supset \forall xA.$ (4, 5)
7. $\vdash_L \forall xA \supset Ey \supset \langle y:x\rangle A.$ (FUI$_s$)
8. $\vdash_L \forall y\forall xA \supset \forall y\langle y:x\rangle A.$ (7, (UG), (UD), ($\forall$Ex))
9. $\vdash_L \forall xA \supset \forall y\forall xA.$ ((VQ), y not free in A)
10. $\vdash_L \forall xA \supset \forall y\langle y:x\rangle A.$ (8, 9)
11. $\vdash_L \forall xA \leftrightarrow \forall y\langle y:x\rangle A.$ (6, 10)

■

Lemma 5.8 (Substituting empty variables)
For any $\mathcal{L}$-sentence A and variables x, y,

(SEV) $\vdash_L x{\neq}x \wedge y{\neq}y \supset (A \leftrightarrow \langle y:x\rangle A).$

Proof (SEV) is trivial if L is positive, in which case $\vdash_L x{=}x$. For negative L, it is proved by induction on A.

1. A is atomic. If $x \notin Var(A)$, then $\vdash_L A \leftrightarrow \langle y:x\rangle A$ by (VS), and so $\vdash_L x{\neq}x \wedge y{\neq}y \supset (A \leftrightarrow \langle y:x\rangle A)$ by (PC). If $x \in Var(A)$, then by (Neg)

   (1) $\vdash_L x{\neq}x \wedge y{\neq}y \supset \neg A.$

   Also by (Neg), $\vdash_L x{\neq}x \wedge y{\neq}y \supset \neg[y/x]A$. By (SAt), $\vdash_L [y/x]A \leftrightarrow \langle y:x\rangle A$, and so $\vdash_L \neg[y/x]A \leftrightarrow \neg\langle y:x\rangle A$. So

   (2) $\vdash_L x{\neq}x \wedge y{\neq}y \supset \neg\langle y:x\rangle A.$

   Combining (1) and (2) yields $\vdash_L x{\neq}x \wedge y{\neq}y \supset (A \leftrightarrow \langle y:x\rangle A)$.

2. A is $\neg B$. By induction hypothesis, $\vdash_L x{\neq}x \wedge y{\neq}y \supset (B \leftrightarrow \langle y:x\rangle B)$. So by (PC), $\vdash_L x{\neq}x \wedge y{\neq}y \supset (\neg B \leftrightarrow \neg\langle y:x\rangle B)$, and by (S$\neg$), $\vdash_L x{\neq}x \wedge y{\neq}y \supset (\neg B \leftrightarrow \langle y:x\rangle\neg B)$.

3. A is $B \supset C$. By induction hypothesis, $\vdash_L x{\neq}x \wedge y{\neq}y \supset (B \leftrightarrow \langle y:x\rangle B)$ and $\vdash_L x{\neq}x \wedge y{\neq}y \supset (C \leftrightarrow \langle y:x\rangle C)$. So by (PC), $\vdash_L x{\neq}x \wedge y{\neq}y \supset ((B \supset C) \leftrightarrow (\langle y:x\rangle B \supset \langle y:x\rangle C))$, and by (S$\supset$), $\vdash_L x{\neq}x \wedge y{\neq}y \supset ((B \supset C) \leftrightarrow \langle y:x\rangle(B \supset C))$.

4. A is $\forall zB$. We distinguish three cases.

a) $z \notin \{x, y\}$. Then

1. $\vdash_L x \neq x \wedge y \neq y \supset (B \leftrightarrow \langle y : x \rangle B)$ (ind. hyp.)
2. $\vdash_L \forall z\, x \neq x \wedge \forall z\, y \neq y \supset (\forall z B \leftrightarrow \forall z \langle y : x \rangle B)$ (1, UG, UD)
3. $\vdash_L x \neq x \wedge y \neq y \supset (\forall z B \leftrightarrow \forall z \langle y : x \rangle B)$ (2, VQ)
4. $\vdash_L x \neq x \wedge y \neq y \supset (\forall z B \leftrightarrow \langle y : x \rangle \forall z B)$. (3, (S$\forall$))

b) $z = x$. Then A is $\forall x B$, and $\vdash_L \forall x B \leftrightarrow \langle y : x \rangle \forall x B$ by (VS). So $\vdash_L x \neq x \wedge y \neq y \supset (\forall x B \leftrightarrow \langle y : x \rangle \forall x B)$ by (PC).

c) $z = y \neq x$. Then A is $\forall y B$. Let v be a variable not in $\mathit{Var}(A), x, y$.

1. $\vdash_L x \neq x \wedge v \neq v \supset (B \leftrightarrow \langle v : x \rangle B)$. (ind. hyp.)
2. $\vdash_L \forall y x \neq x \wedge \forall y v \neq v \supset (\forall y B \leftrightarrow \forall y \langle v : x \rangle B)$. (1, UG, UD)
3. $\vdash_L x \neq x \wedge v \neq v \supset (\forall y B \leftrightarrow \forall y \langle v : x \rangle B)$. (2, VQ)
4. $\vdash_L x \neq x \wedge v \neq v \supset (\forall y B \leftrightarrow \langle v : x \rangle \forall y B)$. (3, (S$\forall$))
5. $\vdash_L \langle y : v \rangle x \neq x \wedge \langle y : v \rangle v \neq v \supset (\langle y : v \rangle \forall y B \leftrightarrow \langle y : v \rangle \langle v : x \rangle \forall y B)$. (4, (Sub$_\mathrm{s}$), (S$\supset$))
6. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y : v \rangle \forall y B \leftrightarrow \langle y : v \rangle \langle v : x \rangle \forall y B)$. (5, (VS), (SAt))
7. $\vdash_L x \neq x \wedge y \neq y \supset (\forall y B \leftrightarrow \langle y : v \rangle \langle v : x \rangle \forall y B)$. (6, (VS))
8. $\vdash_L x \neq x \wedge y \neq y \supset (\forall y B \leftrightarrow \langle y : x \rangle \forall y B)$. (7, (SE2))

5. A is $\langle y_2 : z \rangle B$. We have four cases.

a) $z \notin \{x, y\}$ and $y_2 \neq x$. Then

1. $\vdash_L x \neq x \wedge y \neq y \supset (B \leftrightarrow \langle y : x \rangle B)$ (ind. hyp.)
2. $\vdash_L \langle y_2 : z \rangle x \neq x \wedge \langle y_2 : z \rangle y \neq y \supset (\langle y_2 : z \rangle B \leftrightarrow \langle y_2 : z \rangle \langle y : x \rangle B)$ (1, (Sub$_\mathrm{s}$), (S$\supset$))
3. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y_2 : z \rangle B \leftrightarrow \langle y_2 : z \rangle \langle y : x \rangle B)$ (2, (VS))
4. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y_2 : z \rangle B \leftrightarrow \langle y : x \rangle \langle y_2 : z \rangle B)$. (3, (SS1))

b) $z \neq x$ and $y_2 = x$. Then A is $\langle x : z \rangle B$.

1. $\vdash_L x \neq x \wedge z \neq z \supset (B \leftrightarrow \langle x : z \rangle B)$ (ind. hyp.)
2. $\vdash_L \langle y : z \rangle x \neq x \wedge \langle y : z \rangle z \neq z \supset (\langle y : z \rangle B \leftrightarrow \langle y : z \rangle \langle x : z \rangle B)$ (1, (Sub$_\mathrm{s}$), (S$\supset$))
3. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y : z \rangle B \leftrightarrow \langle y : z \rangle \langle x : z \rangle B)$ (2, (SAt), $z \neq x$)
4. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y : z \rangle B \leftrightarrow \langle x : z \rangle B)$ (3, (VS), $z \neq x$)
5. $\vdash_L x \neq x \wedge y \neq y \supset (B \leftrightarrow \langle y : x \rangle B)$ (ind. hyp.)
6. $\vdash_L \langle y : z \rangle x \neq x \wedge \langle y : z \rangle y \neq y \supset (\langle y : z \rangle B \leftrightarrow \langle y : z \rangle \langle y : x \rangle B)$ (5, (Sub$_\mathrm{s}$),(S$\supset$))
7. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y : z \rangle B \leftrightarrow \langle y : z \rangle \langle y : x \rangle B)$ (6, (SAt), $z \neq x$)
8. $\vdash_L x \neq x \wedge y \neq y \supset (\langle x : z \rangle B \leftrightarrow \langle y : z \rangle \langle y : x \rangle B)$ (4, 7)
9. $\vdash_L x \neq x \wedge y \neq y \supset (\langle x : z \rangle B \leftrightarrow \langle y : x \rangle \langle x : z \rangle B)$. (8, (SS2))

c) $z = x$. Then A is $\langle y_2 : x \rangle B$, and $\vdash_L \langle y_2 : x \rangle B \leftrightarrow \langle y : x \rangle \langle y_2 : x \rangle B$ by (VS). So $\vdash_L x \neq x \wedge y \neq y \supset (\langle y_2 : x \rangle B \leftrightarrow \langle y : x \rangle \langle y_2 : x \rangle B)$ by (PC).

d) $z = y \neq x$ and $y_2 \neq x$. Then A is $\langle y_2 : y\rangle B$. Let v be a variable not in $\mathit{Var}(A), x, y, y_2$.

1. $\vdash_L x \neq x \wedge v \neq v \supset (B \leftrightarrow \langle v : x\rangle B)$. (ind. hyp.)
2. $\vdash_L \langle y_2 : y\rangle x \neq x \wedge \langle y_2 : y\rangle v \neq v \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y_2 : y\rangle\langle v : x\rangle B)$. (1, (Sub$_s$), (S$\supset$))
3. $\vdash_L x \neq x \wedge v \neq v \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y_2 : y\rangle\langle v : x\rangle B)$. (2, (VS))
4. $\vdash_L x \neq x \wedge v \neq v \supset (\langle y_2 : y\rangle B \leftrightarrow \langle v : x\rangle\langle y_2 : y\rangle B)$. (3, (SS1), $y_2 \neq x$)
5. $\vdash_L \langle y : v\rangle x \neq x \wedge \langle y : v\rangle v \neq v \supset (\langle y : v\rangle\langle y_2 : y\rangle B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\langle y_2 : y\rangle B)$. (4, (Sub$_s$), (S$\supset$))
6. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y : v\rangle\langle y_2 : y\rangle B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\langle y_2 : y\rangle B)$. (5, (VS), (SAt))
7. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\langle y_2 : y\rangle B)$. (6, (VS))
8. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : y\rangle B)$. (7, (SE2))

6. A is $\Box B$. Let v be a variable not in $\mathit{Var}(B)$.

1. $\vdash_L x \neq x \wedge v \neq v \supset (B \leftrightarrow \langle v : x\rangle B)$. (ind. hyp.)
2. $\vdash_L \Box(x \neq x \wedge v \neq v) \supset (\Box B \leftrightarrow \Box\langle v : x\rangle B)$. (1, (Nec), (K))
3. $\vdash_L x \neq x \wedge v \neq v \supset \Box(x \neq x \wedge v \neq v)$ ((NA), (EI), (Nec), (K))
4. $\vdash_L x \neq x \wedge v \neq v \supset (\Box B \leftrightarrow \Box\langle v : x\rangle B)$. (2, 3)
5. $\vdash_L x \neq x \wedge v \neq v \supset (\Box B \leftrightarrow \langle v : x\rangle\Box B)$. (4, (S$\Box$), (S$\Diamond$), $v \notin \mathit{Var}(B)$)
6. $\vdash_L \langle y : v\rangle x \neq x \wedge \langle y : v\rangle v \neq v \supset (\langle y : v\rangle\Box B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\Box B)$. (5, (Sub$_s$), (S$\supset$))
7. $\vdash_L x \neq x \wedge y \neq y \supset (\langle y : v\rangle\Box B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\Box B)$. (6, (SAt))
8. $\vdash_L x \neq x \wedge y \neq y \supset (\Box B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\Box B)$. (7, (VS))
9. $\vdash_L x \neq x \wedge y \neq y \supset (\Box B \leftrightarrow \langle y : x\rangle\Box B)$. (8, (SE2))

Now we can prove (SC1) and (SC2). I will also prove that $\langle y : x\rangle A$ and $[y/x]A$ are provably equivalent conditional on $y \neq y$. Compare lemma 3.9 for a (slightly stronger) semantic version of this lemma.

Lemma 5.9 (Substitution conversion)
For any $\mathcal{L}$-formula A and variables x, y,

(SC1) $\vdash_L \langle y : x\rangle A \leftrightarrow [y/x]A$, provided y and x are modally separated in A.

(SC2) $\vdash_L \langle y : x\rangle A \supset [y/x]A$, provided y is modally free for x in A.

(SCN) $\vdash_L y \neq y \supset (\langle y : x\rangle A \leftrightarrow [y/x]A)$.

Proof If x and y are the same variable, then by (SE1), $\vdash_L \langle x : x\rangle A \leftrightarrow [x/x]A$. Assume then that x and y are different variables. We first prove (SC1) and (SC2), by induction on A. Observe that if A is not a box formula $\Box B$, then by definition 3.4, y is modally free for x in A iff y and x are modally separated in A, in which case y and x are also modally separated in any subformula of A.

1. A is atomic. By (SAt), $\vdash_L \langle y : x\rangle A \leftrightarrow [y/x]A$ holds without any restrictions.
2. A is $\neg B$. If y and x are modally separated in A, then by induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by (PC), $\vdash_L \neg\langle y : x\rangle B \leftrightarrow \neg[y/x]B$. By (S$\neg$) and definition 3.3, it follows that $\vdash_L \langle y : x\rangle\neg B \leftrightarrow [y/x]\neg B$.
3. A is $B \supset C$. If y and x are modally separated in A, then by induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$ and $\vdash_L \langle y : x\rangle C \leftrightarrow [y/x]C$. By (S$\supset$), $\vdash_L \langle y : x\rangle(B \supset C) \leftrightarrow (\langle y : x\rangle B \supset \langle y : x\rangle C)$. So $\vdash_L \langle y : x\rangle(B \supset C) \leftrightarrow ([y/x]B \supset [y/x]C)$, and so $\vdash_L \langle y : x\rangle(B \supset C) \leftrightarrow [y/x](B \supset C)$ by definition 3.3.
4. A is $\forall zB$. We have to distinguish four cases, assuming each time that y and x are modally separated in A.
   a) $z \notin \{x, y\}$. By induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by (UG) and (UD), $\vdash_L \forall z\langle y : x\rangle B \leftrightarrow \forall z[y/x]B$. Since $z \notin \{x, y\}$, $\vdash_L \langle y : x\rangle\forall zB \leftrightarrow \forall z\langle y : x\rangle B$ by (S$\forall$), and $\forall z[y/x]B$ is $[y/x]\forall zB$ by definition 3.3; so $\vdash_L \langle y : x\rangle\forall zB \leftrightarrow [y/x]\forall zB$.
   b) $z = y$ and $x \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\forall zB$ is $\forall y[y/x]B$.

1.	$\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$.	(induction hypothesis)
2.	$\vdash_L \forall y\langle y : x\rangle B \leftrightarrow \forall y[y/x]B$.	(1, (UG), (UD))
3.	$\vdash_L B \leftrightarrow \langle y : x\rangle B$.	((VS), $x \notin \mathit{Varf}(B)$)
4.	$\vdash_L \forall yB \leftrightarrow \forall y\langle y : x\rangle B$.	(3, (UG), (UD))
5.	$\vdash_L \forall yB \leftrightarrow \langle y : x\rangle\forall yB$.	((VS), $x \notin \mathit{Varf}(B)$)
6.	$\vdash_L \langle y : x\rangle\forall yB \leftrightarrow \forall y[y/x]B$.	(2, 4, 5)

   c) $z = x$ and $y \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\forall zB$ is $\forall y[y/x]B$.

1.	$\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$.	(induction hypothesis)
2.	$\vdash_L \forall y\langle y : x\rangle B \leftrightarrow \forall y[y/x]B$.	(1, (UG), (UD))
3.	$\vdash_L \forall xB \leftrightarrow \forall y\langle y : x\rangle B$.	((SBV), $y \notin \mathit{Varf}(B)$)
4.	$\vdash_L \forall xB \leftrightarrow \langle y : x\rangle\forall xB$.	(VS)
5.	$\vdash_L \langle y : x\rangle\forall xB \leftrightarrow \forall y[y/x]B$.	(2, 3, 4)

   d) $z = x$ and $y \in \mathit{Varf}(B)$, or $z = y$ and $x \in \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\forall zB$ is $\forall v[y/x][v/z]B$ for some variable $v \notin \mathit{Var}(B) \cup \{x, y\}$. Since v and z are modally separated in B, by induction hypothesis $\vdash_L \langle v : z\rangle B \leftrightarrow [v/z]B$. So by (UG) and (UD), $\vdash_L \forall v\langle v : z\rangle B \leftrightarrow \forall v[v/z]B$. By (SBV), $\vdash_L \forall zB \leftrightarrow \forall v\langle v : z\rangle B$. So $\vdash_L \forall zB \leftrightarrow \forall v[v/z]B$. Moreover, as $z \in \{x, y\}$, y and x are modally separated in $[v/z]B$. So by induction hypothesis, $\vdash_L \langle y:x\rangle[v/z]B \leftrightarrow [y/x][v/z]B$. Then

1.	$\vdash_L \forall zB \leftrightarrow \forall v[v/z]B$	(as just shown)
2.	$\vdash_L \langle y:x\rangle\forall zB \leftrightarrow \langle y:x\rangle\forall v[v/z]B$	(1, (Sub^s), (S$\neg$), (S$\supset$))
3.	$\vdash_L \langle y:x\rangle\forall v[v/z]B \leftrightarrow \forall v\langle y:x\rangle[v/z]B.$	(S$\forall$)
4.	$\vdash_L \langle y:x\rangle\forall zB \leftrightarrow \forall v\langle y:x\rangle[v/z]B.$	(2, 3)
5.	$\vdash_L \langle y:x\rangle[v/z]B \leftrightarrow [y/x][v/z]B.$	(induction hypothesis)
6.	$\vdash_L \forall v\langle y:x\rangle[v/z]B \leftrightarrow \forall v[y/x][v/z]B.$	(5, (UG), (UD))
7.	$\vdash_L \langle y:x\rangle\forall zB \leftrightarrow \forall v[y/x][v/z]B.$	(4, 6)

5. A is $\langle y_2:z\rangle B$. Again we have four cases, assuming x and y are modally separated in A.

a) $z \notin \{x,y\}$. By definition 3.3, then $[y/x]\langle y_2:z\rangle B$ is $\langle[y/x]y_2:z\rangle[y/x]B$.

1.	$\vdash \langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow \langle[y/x]y_2:z\rangle\langle y:x\rangle B$	((SS1) or (SS2))
2.	$\vdash \langle y:x\rangle B \leftrightarrow [y/x]B$	(induction hypothesis)
3.	$\vdash \langle[y/x]y_2:z\rangle(\langle y:x\rangle B \leftrightarrow [y/x]B)$	(2, (Sub$_s$))
4.	$\vdash \langle[y/x]y_2:z\rangle\langle y:x\rangle B \leftrightarrow \langle[y/x]y_2:z\rangle[y/x]B$	(3, (S$\supset$), (S$\neg$))
5.	$\vdash \langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow \langle[y/x]y_2:z\rangle[y/x]B.$	(1, 4)

b) $z = y$ and $x \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\langle y_2:z\rangle B$ is $\langle[y/x]y_2:y\rangle[y/x]B$. By induction hypothesis, $\vdash_L \langle y:x\rangle B \leftrightarrow [y/x]B$. So by (Sub$_s$) and (S$\supset$), $\vdash_L \langle[y/x]y_2:y\rangle\langle y:x\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle[y/x]B$. If $y_2 = x$, then $\vdash_L \langle y:x\rangle\langle y_2:y\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle\langle y:x\rangle B$ by (SS2). If $y_2 \neq x$, then

1.	$\vdash_L \langle y_2:y\rangle B \leftrightarrow \langle y:x\rangle\langle y_2:y\rangle B$	((VS), $x \notin \mathit{Varf}(\langle y_2:y\rangle B)$)
2.	$\vdash_L B \leftrightarrow \langle y:x\rangle B$	((VS), $x \notin \mathit{Varf}(B)$)
3.	$\vdash_L \langle y_2:y\rangle B \leftrightarrow \langle y_2:y\rangle\langle y:x\rangle B$	(1, (Sub$_s$), (S$\supset$))
4.	$\vdash_L \langle y:x\rangle\langle y_2:y\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle\langle y:x\rangle B$	(1, 3)

So either way $\vdash_L \langle y:x\rangle\langle y_2:y\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle\langle y:x\rangle B$. So $\vdash_L \langle y:x\rangle\langle y_2:y\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle[y/x]B$.

c) $z = x$ and $y \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\langle y_2:z\rangle B$ is $\langle[y/x]y_2:y\rangle[y/x]B$. By induction hypothesis, $\vdash_L \langle y:x\rangle B \leftrightarrow [y/x]B$. So by (Sub$_s$) and (S$\supset$), $\vdash_L \langle[y/x]y_2:y\rangle\langle y:x\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle[y/x]B$. Since $y \notin \mathit{Varf}(B)$, by (SE2), $\vdash_L \langle[y/x]y_2:y\rangle\langle y:x\rangle B \leftrightarrow \langle[y/x]y_2:x\rangle B$. Moreover, $\vdash_L \langle[y/x]y_2:x\rangle B \leftrightarrow \langle y:x\rangle\langle y_2:x\rangle B$ by either (VS) (if $x \neq y_2$) or by (SE1), (Sub$_s$) and (S$\supset$) (if $x = y_2$). So $\vdash_L \langle y:x\rangle\langle y_2:x\rangle B \leftrightarrow \langle[y/x]y_2:y\rangle\langle y:x\rangle B$.

d) $z = x$ and $y \in \mathit{Varf}(B)$, or $z = y$ and $x \in \mathit{Varf}(B)$. By definition 3.3, then

$[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : v\rangle [y/x][v/z]B$, where $v \notin \mathit{Var}(B) \cup \{x, y, y_2\}$.

1.	$\vdash \langle v : z\rangle B \leftrightarrow [v/z]B$	(induction hypothesis)
2.	$\vdash \langle y_2 : v\rangle\langle v : z\rangle B \leftrightarrow \langle y_2 : v\rangle[v/z]B$	(1, (Sub$_\mathrm{s}$), (S$\supset$), (S$\neg$))
3.	$\vdash \langle y_2 : z\rangle B \leftrightarrow \langle y_2 : v\rangle\langle v : z\rangle B$	(SE2)
4.	$\vdash \langle y_2 : z\rangle B \leftrightarrow \langle y_2 : v\rangle[v/z]B$	(2, 3)

Since $z \in \{x, y\}$, x and y are modally separated in $[v/z]B$. So:

5.	$\vdash \langle y : x\rangle[v/z]B \leftrightarrow [y/x][v/z]B$	(ind. hyp.)
6.	$\vdash \langle [y/x]y_2 : v\rangle\langle y : x\rangle[v/z]B \leftrightarrow \langle [y/x]y_2 : v\rangle[y/x][v/z]B$	(5, (Sub$_\mathrm{s}$), (S$\supset$))
7.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : v\rangle[v/z]B$	(4, (Sub$_\mathrm{s}$), (S$\supset$))
8.	$\vdash \langle y : x\rangle\langle y_2 : v\rangle[v/z]B \leftrightarrow \langle [y/x]y_2 : v\rangle\langle y : x\rangle[v/z]B$	((SS1) or (SS2))
9.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : v\rangle\langle y : x\rangle[v/z]B$	(7, 8)
10.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : v\rangle[y/x][v/z]B$	(6, 9)

6. A is $\Box B$. For (SC1), assume x and y are modally separated in A. Then they are also modally separated in B, so by induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. By (Nec) and (K), then $\vdash_L \Box\langle y : x\rangle B \leftrightarrow \Box[y/x]B$. By (S$\Box$), $\vdash_L \langle y : x\rangle\Box B \supset \Box\langle y : x\rangle B$. Since at most one of x, y is free in B, by (S$\Diamond$), $\vdash_L \langle y : x\rangle\Diamond\neg B \supset \Diamond\langle y : x\rangle\neg B$; so $\vdash_L \Box\langle y : x\rangle B \supset \langle y : x\rangle\Box B$ (by (S$\neg$), (Sub$_\mathrm{s}$), (S$\supset$), (Nec), (K)). So $\vdash_L \langle y : x\rangle\Box B \leftrightarrow \Box[y/x]B$. Since $\Box[y/x]B$ is $[y/x]\Box B$ by definition 3.3, this means that $\vdash_L \langle y : x\rangle\Box B \leftrightarrow [y/x]\Box B$.

   For (SC2), assume y is modally free for x in $\Box B$. Then y is modally free for x in B, so by induction hypothesis, $\vdash \langle y : x\rangle B \supset [y/x]B$. By (Nec) and (K), then $\vdash \Box\langle y : x\rangle B \supset \Box[y/x]B$. By (S$\Box$), $\vdash \langle y : x\rangle\Box B \supset \Box\langle y : x\rangle B$. So $\vdash \langle y : x\rangle\Box B \supset \Box[y/x]B$.

Here is the proof for (SCN). The first three clauses are very similar.

1. A is atomic. Then $\vdash_L \langle y : x\rangle A \leftrightarrow [y/x]A$ as we've seen above, and so $\vdash_L y \neq y \supset (\langle y : x\rangle A \leftrightarrow [y/x]A)$ by (PC).

2. A is $\neg B$. By induction hypothesis, $\vdash_L y \neq y \supset (\langle y : x\rangle B \leftrightarrow [y/x]B)$. So by (PC), $\vdash_L y \neq y \supset (\neg\langle y : x\rangle B \leftrightarrow \neg[y/x]B)$. By (S$\neg$) and definition 3.3, it follows that $\vdash_L y \neq y \supset (\langle y : x\rangle\neg B \leftrightarrow [y/x]\neg B)$.

3. A is $B \supset C$. By induction hypothesis, $\vdash_L y \neq y \supset (\langle y : x\rangle B \leftrightarrow [y/x]B)$ and $\vdash_L y \neq y \supset (\langle y : x\rangle C \leftrightarrow [y/x]C)$. By (S$\supset$), $\vdash_L y \neq y \supset (\langle y : x\rangle(B \supset C) \leftrightarrow (\langle y : x\rangle B \supset \langle y : x\rangle C))$. So $\vdash_L y \neq y \supset (\langle y : x\rangle(B \supset C) \leftrightarrow ([y/x]B \supset [y/x]C))$, and so $\vdash_L y \neq y \supset (\langle y : x\rangle(B \supset C) \leftrightarrow [y/x](B \supset C))$ by definition 3.3.

4. A is $\forall zB$. If $z \notin \{x, y\}$, then by induction hypothesis, $\vdash_L y \neq y \supset (\langle y : x\rangle B \leftrightarrow [y/x]B)$. So by (UG) and (UD), $\vdash_L \forall z\, y \neq y \supset (\forall z\langle y : x\rangle B \leftrightarrow \forall z[y/x]B)$. Since $z \notin \{x, y\}$, $\vdash_L \langle y : x\rangle\forall zB \leftrightarrow \forall z\langle y : x\rangle B$ by (S$\forall$), and $\vdash_L y \neq y \supset \forall z\, y \neq y$ by (VQ), and $\forall z[y/x]B$ is $[y/x]\forall zB$ by definition 3.3; so $\vdash_L y \neq y \supset (\langle y : x\rangle\forall zB \leftrightarrow [y/x]\forall zB)$.

   Alternatively, if $z \in \{x, y\}$, then either x or y is not free in A, and thus x and y are modally separated in A. By (SC2), then $\vdash_L \langle y : x\rangle\forall zB \leftrightarrow [y/x]\forall zB$, and so by (PC), $\vdash_L y \neq y \supset (\langle y : x\rangle\forall zB \leftrightarrow [y/x]\forall zB)$.

5. A is $\langle y_2 : z\rangle B$. If $z \notin \{x, y\}$, then by induction hypothesis, $\vdash_L y \neq y \supset (\langle y : x\rangle B \leftrightarrow [y/x]B)$. So by (Sub$_\mathrm{s}$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : z\rangle y \neq y \supset (\langle [y/x]y_2 : z\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle [y/x]B)$. By (VS), $\langle [y/x]y_2 : z\rangle y \neq y \leftrightarrow y \neq y$. And by (SS1) or (SS2), $\langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle\langle y : x\rangle B$. So $\vdash_L y \neq y \supset (\langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle [y/x]B)$. But by definition 3.3, $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : y\rangle [y/x]B$.

   Alternatively, if $z \in \{x, y\}$, then either x or y is not free in A, and thus x and y are modally separated in A. By (SC2), then $\vdash_L \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow [y/x]\langle y_2 : z\rangle B$, and so by (PC), $\vdash_L y \neq y \supset (\langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow [y/x]\langle y_2 : z\rangle B)$.

6. A is $\Box B$. Then

1.	$\vdash_L y \neq y \supset (\langle y : x\rangle B \leftrightarrow [y/x]B)$.	(ind. hyp.)
2.	$\vdash_L \Box y \neq y \supset (\Box\langle y : x\rangle B \leftrightarrow \Box[y/x]B)$.	(1, (Nec), (K))
3.	$\vdash_L y \neq y \supset \Box y \neq y$.	((=R) or (NA), (EI) and (Nec))
4.	$\vdash_L y \neq y \supset (\Box\langle y : x\rangle B \leftrightarrow \Box[y/x]B)$.	(2, 3)
5.	$\vdash_L y \neq y \supset \langle y : x\rangle(x \neq x \wedge y \neq y)$	((SAt), (S$\supset$), (S$\neg$))
6.	$\vdash_L (x \neq x \wedge y \neq y) \supset \Box(x \neq x \wedge y \neq y)$.	((=R) or (NA), (EI), (Nec) and
7.	$\vdash_L \Box(x \neq x \wedge y \neq y) \supset (\Box B \leftrightarrow \Box\langle y : x\rangle B)$.	((SEV), (Nec), (K))
8.	$\vdash_L (x \neq x \wedge y \neq y) \supset (\Box B \leftrightarrow \Box\langle y : x\rangle B)$.	(6, 7)
9.	$\vdash_L \langle y : x\rangle(x \neq x \wedge y \neq y) \supset (\langle y : x\rangle\Box B \leftrightarrow \langle y : x\rangle\Box\langle y : x\rangle B)$.	(8, (Sub$_\mathrm{s}$), (S$\supset$))
10.	$\vdash_L \langle y : x\rangle(x \neq x \wedge y \neq y) \supset (\langle y : x\rangle\Box B \leftrightarrow \Box\langle y : x\rangle B)$.	(9, (VS))
11.	$\vdash_L y \neq y \supset (\langle y : x\rangle\Box B \leftrightarrow \Box\langle y : x\rangle B)$.	(7, 10)
12.	$\vdash_L y \neq y \supset (\langle y : x\rangle\Box B \leftrightarrow [y/x]\Box B)$.	(4, 13, def. 3.3)

■

Lemma 5.10 (Syntactic alpha-conversion)
If A, A' are $\mathcal{L}$-formulas, and A' is an alphabetic variant of A, then

(AC) $\vdash_L A \leftrightarrow A'$.

Proof by induction on A.

1. A is atomic. Then $A = A'$ and $\vdash_L A \leftrightarrow A'$ by (Taut).

2. A is $\neg B$. Then A' is $\neg B'$ with B' an alphabetic variant of B. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. By (PC), $\vdash_L \neg B \leftrightarrow \neg B'$.

3. A is $B \supset C$. Then A' is $B' \supset C'$ with B', C' alphabetic variants of B, C, respectively. By induction hypothesis, $\vdash_L B \leftrightarrow B'$ and $\vdash_{sC} C \leftrightarrow C'$. By (PC), then $\vdash_L (B \supset C) \leftrightarrow (B' \supset C')$.

4. A is $\forall x B$. Then A' is either $\forall x B'$ or $\forall z[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that A' is $\forall x B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (UG) and (UD), $\vdash_L \forall x B \leftrightarrow \forall x B'$.

   Alternatively, assume A' is $\forall z[z/x]B'$ and $z \notin \mathit{Var}(B')$. Since B' differs from B at most in renaming bound variables, if z were free in B, then $z \in \mathit{Var}(B')$. So z is not free in B. Then

1.	$\vdash_L B \leftrightarrow B'$.	induction hypothesis
2.	$\vdash_L \langle z : x\rangle B \leftrightarrow \langle z : x\rangle B'$.	(1, $(\mathrm{Sub_s})$, $(\mathrm{S}\neg)$)
3.	$\vdash_L \langle z : x\rangle B' \leftrightarrow [z/x]B'$.	((SC1), $z \notin \mathit{Var}(B')$)
4.	$\vdash_L \langle z : x\rangle B \leftrightarrow [z/x]B'$.	(2, 3)
5.	$\vdash_L \forall z\langle z : x\rangle B \leftrightarrow \forall z[z/x]B'$.	(4, (UG), (UD))
6.	$\vdash_L \forall x B \leftrightarrow \forall z\langle z : x\rangle B$.	((SBV), z not free in B)
7.	$\vdash_L \forall x B \leftrightarrow \forall z[z/x]B'$.	(5, 6)

5. A is $\langle y : x\rangle B$. Then A' is either $\langle y : x\rangle B'$ or $\langle y : z\rangle[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B)$. Assume first that A' is $\langle y : x\rangle B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by $(\mathrm{Sub_s})$ and $(\mathrm{S}\supset)$, $\vdash_L \langle y : x\rangle B \leftrightarrow \langle y : x\rangle B'$.

   Alternatively, assume A' is $\langle y : z\rangle[z/x]B'$ and $z \notin \mathit{Var}(B')$. Again, it follows that z is not free in B. So

1.	$\vdash_L B \leftrightarrow B'$.	induction hypothesis
2.	$\vdash_L \langle z : x\rangle B \leftrightarrow \langle z : x\rangle B'$.	(1, $(\mathrm{Sub_s})$, $(\mathrm{S}\supset)$)
3.	$\vdash_L \langle z : x\rangle B' \leftrightarrow [z/x]B'$.	((SC1), $z \notin \mathit{Var}(B')$)
4.	$\vdash_L \langle z : x\rangle B \leftrightarrow [z/x]B'$.	(2, 3)
5.	$\vdash_L \langle y : z\rangle\langle z : x\rangle B \leftrightarrow \langle y : z\rangle[z/x]B'$.	(4, $(\mathrm{Sub_s})$, $(\mathrm{S}\supset)$)
6.	$\vdash_L \langle y : z\rangle\langle z : x\rangle B \leftrightarrow \langle y : x\rangle B$.	((SE2), z not free in B)
7.	$\vdash_L \langle y : x\rangle B \leftrightarrow \langle y : z\rangle[z/x]B'$.	(5, 6)

6. A is $\Box A'$. Then B is $\Box B'$ with B' an alphabetic variant of A'. By induction hypothesis, $\vdash_L A' \leftrightarrow B'$. Then by (Nec), $\vdash_L \Box(A' \leftrightarrow B')$, and by (K), $\vdash_L \Box A' \leftrightarrow \Box B'$. ■

Theorem 5.11 (Substitution and non-substitution logics)
For any $\mathcal{L}$-formula A and variables x, y,

(FUI*) $\vdash_L \forall x A \supset (Ey \supset [y/x]A)$, provided y is modally free for x in A,

(LL*) $\vdash_L x = y \supset A \supset [y/x]A$, provided y is modally free for x in A,

(Sub*) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is modally free for x in A.

It follows that $\mathsf{P} \subseteq \mathsf{P}_s$ and $\mathsf{N} \subseteq \mathsf{N}_s$.

Proof Assume y is modally free for x in A. Then by (SC2), $\vdash_L \langle y : x\rangle A \supset [y/x]A$. By ($\mathrm{FUI_s}$), $\vdash_L \forall x A \supset (Ey \supset \langle y : x\rangle A)$, so by (PC), $\vdash_L \forall x A \supset (Ey \supset [y/x]A)$. Similarly, by ($\mathrm{LL_s}$), $\vdash_L x = y \supset A \supset \langle y : x\rangle A$, so by (PC), $\vdash_L x = y \supset A \supset [y/x]A$. Finally, by ($\mathrm{Sub_s}$), if $\vdash_L A$, then $\vdash_L \langle y : x\rangle A$, so then $\vdash_L [y/x]A$ by (PC). ■

Lemma 5.12 (Symmetry and transitivity of identity)
For any $\mathcal{L}$-variables x, y, z,

(=S) $\vdash_L x = y \supset y = x$;

(=T) $\vdash_L x = y \supset y = z \supset x = z$.

Proof Immediate from lemma 5.11 and lemma 4.10. ■

Lemma 5.13 (Variations on Leibniz' Law)
If A is an $\mathcal{L}$-formula and x, y, y' are $\mathcal{L}$-variables, then

(LV1) $\vdash_L x = y \supset \langle y : x\rangle A \supset A$.

(LV2) $\vdash_L y = y' \supset \langle y : x\rangle A \supset [y'/x]A$, provided y' is modally free for x in A.

Proof (LV1). Let z be an $\mathcal{L}$-variable not in $\mathit{Var}(A)$. Then

1.	$\vdash_L x = z \supset \langle z : x\rangle A \supset \langle x : z\rangle\langle z : x\rangle A$.	($\mathrm{LL_s}$)
2.	$\vdash_L x = z \supset \langle z : x\rangle A \supset \langle x : x\rangle A$.	(1, (SE2), $z \notin \mathit{Var}(A)$)
3.	$\vdash_L x = z \supset \langle z : x\rangle A \supset A$.	(2, (SE1))
4.	$\vdash_L \langle y : z\rangle x = z \supset \langle y : z\rangle\langle z : x\rangle A \supset \langle y : z\rangle A$.	(3, (VS), (S$\supset$))
5.	$\vdash_L x = z \supset \langle y : z\rangle\langle z : x\rangle A \supset \langle y : z\rangle A$.	(4, (SAt))
6.	$\vdash_L x = z \supset \langle y : x\rangle A \supset \langle y : z\rangle A$.	(5, (SE2), $z \notin \mathit{Var}(A)$)
7.	$\vdash_L x = z \supset \langle y : x\rangle A \supset A$.	(6, (VS), $z \notin \mathit{Var}(A)$).

(LV2).

1.	$\vdash_L x{=}y \wedge y{=}y' \supset x{=}y'$.	$({=}\mathrm{T})$
2.	$\vdash_L A \wedge x{=}y' \supset [y'/x]A$.	((LL*), y' m.f. in A)
3.	$\vdash_L A \wedge x{=}y \wedge y{=}y' \supset [y'/x]A$.	(1, 2)
4.	$\vdash_L \langle y : x\rangle A \wedge \langle y : x\rangle x{=}y \wedge \langle y : x\rangle y{=}y' \supset \langle y : x\rangle [y'/x]A$.	(3, (Sub$_\mathrm{s}$), (S$\neg$), (S$\supset$))
5.	$\vdash_L y{=}y \supset \langle y : x\rangle x{=}y$.	(SAt)
6.	$\vdash_L y{=}y' \supset y{=}y$.	((LL*), $({=}\mathrm{S})$)
7.	$\vdash_L y{=}y' \supset \langle y : x\rangle y{=}y'$.	(VS)
8.	$\vdash_L \langle y : x\rangle A \wedge y{=}y' \supset \langle y : x\rangle [y'/x]A$.	(4, 5, 6, 7)
9.	$\vdash_L \langle y : x\rangle [y'/x]A \supset [y'/x]A$.	(VS)
10.	$\vdash_L \langle y : x\rangle A \wedge y{=}y' \supset [y'/x]A$.	(8, 9)

■

Lemma 5.14 (Leibniz' Law with sequences)
For any $\mathcal{L}$-formula A and variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(LL$_\mathrm{n}$) $\vdash_L x_1{=}y_1 \wedge \ldots \wedge x_n{=}y_n \supset A \supset \langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$.

Proof For $n = 1$, (LL$_\mathrm{n}$) is (LL$_\mathrm{s}$). Assume then that $n > 1$. To keep formulas in the following proof at a managable length, let $\underline{\phi(i)}$ abbreviate the sequence $\phi(1), \ldots, \phi(n-1)$. For example, $\langle \underline{y_i} : \underline{x_i}\rangle$ is $\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle$. Let z be the alphabetically first variable not in A or

$x_1, \ldots, x_n$. Now

1. $\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A.$ (LL$_\mathrm{s}$)
2. $\vdash_L \langle y_n : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A.$ (SE1)
3. $\vdash_L \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ ((SS1) or (SS2))
4. $\vdash_L \langle y_n : z \rangle \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A$
   $\supset \langle y_n : z \rangle \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ (3, (Sub$_\mathrm{s}$), (S$\supset$))
5. $\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ (1, 2, 4)
6. $\vdash_L x_n = z \supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$
   $\supset \langle z : x_n \rangle \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ (LL$_\mathrm{s}$)
7. $\vdash_L x_n = z \supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$
   $\supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$ (6, (SS1))
8. $\vdash_L z = x_n \supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A$
   $\supset \langle x_n : z \rangle \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$ (LL$_\mathrm{s}$)
9. $\vdash_L z = x_n \supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A$
   $\supset \langle \underline{y_i} : \underline{x_i} \rangle \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$ (8, (SS1), (SS2))
10. $\vdash_L \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A \leftrightarrow \langle z : x_n \rangle A$ ((SE1), (SE2))
11. $\vdash_L \langle \underline{y_i} : \underline{x_i} \rangle \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A \supset \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ (10, (Sub$_\mathrm{s}$), (S$\supset$))
12. $\vdash_L z = x_n \supset x_n = z$ (=S)
13. $\vdash_L z = x_n \supset \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A \supset \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ (7, 9, 11, 12)
14. $\vdash_L x_n = y_n \supset \langle y_n : z \rangle z = x_n$ ((=S), (SAt))
15. $\vdash_L x_n = y_n \supset \langle y_n : z \rangle \langle \underline{[z/x_n]y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$
    $\supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ 13, 14, (Sub$_\mathrm{s}$), (S$\supset$)
16. $\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ 5, 15
17. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset A \supset \langle \underline{y_i} : \underline{x_i} \rangle A.$ (induction hypothesis)
18. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$ (16, 17)
19. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A.$ (18, def. 3.14)

■

Lemma 5.15 (Closure under transformations)
For any $\mathcal{L}$-formula A and transformation τ on $\mathcal{L}$,

(Sub$^\tau$) $\vdash_L A$ iff $\vdash_L A^\tau$.

Proof The proof is exactly as in lemma 4.12. ■

6 Correspondence

A well-known feature of Kripke semantics for propositional modal logic is that various modal principles correspond to conditions on the accessibility relation, in the sense that the principle is valid in all and only those Kripke frames whose accessibility relation satisfies the condition: $\Box p \supset p$ *corresponds to* (or *defines*) the class of reflexive frames, $p \supset \Box\Diamond p$ the class of symmetrical frames, and so on.

In our counterpart semantics, all these facts are preserved if we translate the sentence letters of propositional modal logic into null-ary predicates (or other sentences without free variables). Things are slightly more complex if we substitute the sentence letters by formulas with free variables. For example, the general schema $\Box A \supset A$, where A may contain arbitrarily many free variables, is valid in a counterpart structure iff (i) every world can see itself, and (ii) every individual (and every sequence of individuals) at every world is a counterpart of itself (relative to some counterpart relation). To see why (i) is not enough, consider what is required for the validity of $\Box Fx \supset Fx$. Loosely speaking, the antecedent $\Box Fx$ is true at w iff all counterparts of x are F at all accessible worlds. This does not entail that x is F at w unless (i) w can see itself and (ii) x is its own counterpart at w.

In a sense, formulas containing free variables don't just express properties of worlds, but relations between a world and a (finite) sequence of individuals. In any given counterpart model, the truth-value of Fx is fixed by choosing a world. In *this* sense, the formula expresses just a property of worlds. On the other hand, the truth-value of $\forall xFx$ (or $\Box Fx$) at a world is not simply a function of the truth-value of Fx at that world or other worlds. Whether $\forall xFx$ is true at w depends on whether Fx is true at w relative to every choice of an individual $d \in D_w$ as value of x. (Similarly, whether $\Box Fx$ is true at w depends on whether Fx is true at all accessible worlds w' relative to every choice of an x-counterpart as value of x.) In a Tarski-style semantics this is rendered explicit by the fact that formulas with free variables are evaluated not only relative to a world, but relative to a world w and an infinite sequence of individuals from w, representing different assignments of values to the variables. (Our Mates-style semantics incorporates the assignment function in the interpretation function V; hence our rules for quantifiers quantify over alternative interpretations rather than alternative assignments.) That's the sense in which formulas express properties of sequences. Tarski's semantics misleadingly suggests that relevant sequences are infinite, although actual formulas of standard first-order logic only contain a finite number of variables and thus only ever constrain a finite initial segment of a Tarskian sequence.

First a brief review of some definitions from propositional modal logic.

Definition 6.1 (Languages of propositional modal logic)
A set of formulas $\mathcal{L}_0$ is a *(unimodal) propositional language* if there is a denumerable set of symbols *Prop* (the *sentence letters* of $\mathcal{L}_0$) distinct from $\{\neg, \supset, \Box\}$ such that $\mathcal{L}_0$ is generated by the rule

$$P \mid \neg A \mid (A \supset B) \mid \Box A,$$

where $P \in \mathit{Prop}$.

Note that any language $\mathcal{L}$ of quantified modal logic is also a unimodal propositional language, with sentence letters defined as all $\mathcal{L}$-formulas not of the form $\neg A$, $A \supset B$ or $\Box A$. (So $\forall x \Box(Fx \supset Gx)$, for example, is a sentence letter.) For future reference, let's call such formulas *quasi-atomic*.

Definition 6.2 (Quasi-atomic formulas)
A formula A of quantified modal logic is *quasi-atomic* if it is not of the form $\neg B$, $B \supset C$ or $\Box B$.

Definition 6.3 (Frames and valuations)
A *frame* is a pair consisting of a non-empty set W and a relation $R \subseteq W^2$.

A *valuation of* a unimodal propositional language $\mathcal{L}_0$ *on* a frame $\mathcal{F} = \langle W, R \rangle$ is a function V that maps every sentence letter of $\mathcal{L}_0$ to a subset of W.

Definition 6.4 (Propositional truth)
For any frame $\mathcal{F} = \langle W, R \rangle$, point $w \in W$, and valuation V on $\mathcal{F}$,

$$\begin{array}{ll}
\mathcal{F}, V, w \Vdash_K P & \text{for } P \in \mathit{Prop} \text{ iff } w \in V(P), \\
\mathcal{F}, V, w \Vdash_K \neg B & \text{iff } \mathcal{F}, V, w \nVdash_K B, \\
\mathcal{F}, V, w \Vdash_K B \supset C & \text{iff } \mathcal{F}, V, w \nVdash_K B \text{ or } \mathcal{F}, V, w \Vdash_K C, \\
\mathcal{F}, V, w \Vdash_K \Box B & \text{iff } \mathcal{F}, V, w' \Vdash_K B \text{ for all } w' \text{ with } wRw'.
\end{array}$$

Definition 6.5 (Frame validity)
A formula A of a unimodal propositional language is *valid in a frame* $\mathcal{F} = \langle W, R\rangle$ if $\mathcal{F}, V, w \Vdash_K A$ for all $w \in W$ and valuation functions V of the language on $\mathcal{F}$. A set of formulas $\mathbb{A}$ is *valid in a frame* $\mathcal{F}$ if all members of $\mathbb{A}$ are valid in $\mathcal{F}$.

To keep things simple, I will focus on positive models for a moment.

Definition 6.6 (n-sequential accessibility relations)
Given a total counterpart structure $\mathcal{S} = \langle W, R, U, D, K\rangle$ and number $n \in \mathbb{N}$, the *n-sequential accessibility relation $R^n_{\mathcal{S}}$ of $\mathcal{S}$* is the binary relation such that $\langle w, d_1, \ldots, d_n\rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n\rangle$ iff wRw' and for some $C \in K_{w,w'}$, $d_1 C d'_1, \ldots, d_n C d'_n$.

(Note that $R^0_{\mathcal{S}} = R$.) We can reformulate the semantics of section 2 in terms of $R^n_{\mathcal{S}}$.

Lemma 6.7 (Sequential semantics)
Let $A(x_1, \ldots, x_n)$ be a formula with free variables $x_1, \ldots, x_n$. Restricted to total structures $\mathcal{S}$ and interpretations V, the clause for the box in definition 2.7, viz.

$$w, V \Vdash_{\mathcal{S}} \Box A(x_1, \ldots, x_n) \text{ iff } w', V' \Vdash_{\mathcal{S}} A \text{ for all } w', V' \text{ such that } wRw' \text{ and } V_w \triangleright V'_{w'}.$$

is equivalent to

$$w, V \Vdash_{\mathcal{S}} \Box A(x_1, \ldots, x_n) \text{ iff } w', V' \Vdash_{\mathcal{S}} A(x_1, \ldots, x_n) \text{ for all } w', V' \text{ such that } \langle w, V_w(x_1), \ldots, V_w(x_n)\rangle R^n_{\mathcal{S}} \langle w', V'_{w'}(x_1,) \ldots, V'_{w'}(x_n)\rangle.$$

Proof By definitions 2.7 and 2.6, $w, V \Vdash_{\mathcal{S}} \Box A(x_1, \ldots, x_n)$ iff

(1) $w', V' \Vdash_{\mathcal{S}} A(x_1, \ldots, x_n)$ for all w', V' such that wRw' and for some $C \in K_{w,w'}$ and all variables x, $V_w(x)$ is C-related to $V'_{w'}(x)$.

By lemma 2.10, it doesn't matter what V' assigns to variables not in $A(x_1, \ldots, x_n)$. So (1) is equivalent to

(2) $w', V' \Vdash_{\mathcal{S}} A(x_1, \ldots, x_n)$ for all w', V' such that wRw' and for some $C \in K_{w,w'}$ and all variables $x_i \in x_1, \ldots, x_n$, $V_w(x_i)$ is C-related to $V'_{w'}(x_i)$.

By definition 6.6, $\langle w, V_w(x_1), \ldots, V_w(x_n)\rangle R^n_{\mathcal{S}} \langle w', V'_{w'}(x_1), \ldots, V_w(x_n)\rangle$ iff wRw' and for some $C \in K_{w,w'}$ and all variables $x_i \in x_1, \ldots, x_n$, $V_w(x_i)$ is C-related to $V'_{w'}(x_i)$. So (2) is equivalent to

(3) $w', V' \Vdash_{\mathcal{S}} A(x_1, \ldots, x_n)$ for all w', V' such that $\langle w, V_w(x_1), \ldots, V_w(x_n) \rangle R^n_{\mathcal{S}} \langle w', V'_{w'}(x_1), \ldots, V'_{w'}(x_n) \rangle$.

■

By lemma 2.9, the truth-value of $A(x_1, \ldots, x_n)$ at a world w in a model $\mathcal{S}, V$ never depends on the values V assigns to variables other than $x_1, \ldots, x_n$, nor on the values V assigns to $x_1, \ldots, x_n$ relative to worlds other than w. Thus we can factor V into a predicate interpretation I and the sequence $V_w(x_1), \ldots, v_w(x_n)$ specifying the values assigned to $x_1, \ldots, x_n$ at w, and once again reformulate our semantics.

Definition 6.8 (Rank)
Let ρ be some fixed "alphabetical" order on the variables of $\mathcal{L}$, i.e. a bijection $Var \rightarrow \mathbb{N}^+$. I will use v to denote the inverse of ρ, so that v_1 is the alphabetically first variable, v_2 the second, and so on. The *rank* of an $\mathcal{L}$-formula A is the smallest number $r \in \mathbb{N}$ such that all members of $Var(A)$ have a ρ-value less than r.

Definition 6.9 (Finitary satisfaction)
Let A be an $\mathcal{L}$-formula and r a number greater or equal to A's rank. Let $\mathcal{S} = \langle W, R, U, D, K \rangle$ be a total counterpart structure, I a predicate interpretation on $\mathcal{S}$, w a member of W, and $d_1, \ldots, d_r$ (not necessarily distinct) elements of U_w. Then

$w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} Px_1 \ldots x_n$	iff $\langle d_{\rho(x_1)}, \ldots, d_{\rho(x_n)} \rangle \in I_w(P)$.
$w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \neg A$	iff $w, d_1, \ldots, d_r \nVvdash_{\mathcal{S},I} A$.
$w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} A \supset B$	iff $w, d_1, \ldots, d_r \nVvdash_{\mathcal{S},I} A$ or $w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} B$.
$w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \forall x A$	iff $w, d'_1, \ldots, d'_r \Vvdash_{\mathcal{S},I} A$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} \in D_w$ and $d'_i = d_i$ for all $i \neq \rho(x)$.
$w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \langle y : x \rangle A$	iff $w, d'_1, \ldots, d'_r \Vvdash_{\mathcal{S},I} A$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} = d_{\rho(y)}$ and $d'_i = d_i$ for all $i \neq \rho(x)$.
$w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \Box A$	iff $w', d'_1, \ldots, d'_r \Vvdash_{\mathcal{S},I} A$ for all $w, d'_1, \ldots, d'_r$ such that $\langle w, d_1, \ldots, d_r \rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n \rangle$,

where $R^n_{\mathcal{S}}$ is the n-sequential counterpart relation of $\mathcal{S}$.

Notice that this looks just like standard Kripke semantics for a propositional modal language with multiple box operators $\Box$, $\forall v_1, \ldots, \forall v_r$, $\langle v_1 : v_1 \rangle, \langle v_1 : v_2 \rangle, \ldots, \langle v_r : v_r \rangle$, all governed by their own accessibility relation between points of the form $w, d_1, \ldots, d_r$. (As mentioned on p. 16, there is a bit of redundancy here: if we have substitution operators, a single box operator $\forall v_1$ would be enough.)

LEMMA 6.10 (TRUTH AND SATISFACTION)
For any total counterpart structure $\mathcal{S} = \langle W, R, U, D, K \rangle$, total interpretation V on $\mathcal{S}$, world $w \in W$, and formula A,

$$w, V \Vdash_{\mathcal{S}} A \text{ iff } w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} A,$$

where I is V restricted to predicates, r is a number greater than or equal to A's rank and $d_1 = V_w(v_1), \ldots, d_r = V_w(v_r)$.

PROOF by induction on A.

(i) A is $Px_1 \ldots x_n$. $w, V \Vdash_{\mathcal{S}} Px_1 \ldots x_n$ iff $\langle V_w(x_1), \ldots, V_w(x_n) \rangle \in V_w(P)$ by definition 2.7, iff $\langle d_{\rho(x_1)}, \ldots, d_{\rho(x_n)} \rangle \in I_w(P)$, iff $x, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} Px_1 \ldots x_n$ by definition 6.9.

(ii) A is $\neg B$. $w, V \Vdash_{\mathcal{S}} \neg B$ iff $w, V \nVdash_{\mathcal{S}} B$ by definition 2.7, iff $w, d_1, \ldots, d_r \not\Vvdash_{\mathcal{S},I} B$ by induction hypothesis (since B has rank $\leq r$), iff $w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \neg B$ by definition 6.9.

(iii) A is $B \supset C$. $w, V \Vdash_{\mathcal{S}} B \supset C$ iff $w, V \nVdash_{\mathcal{S}} B$ or $w, V \Vdash_{\mathcal{S}} C$ by definition 2.7, iff $w, d_1, \ldots, d_r \not\Vvdash_{\mathcal{S},I} B$ or $w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} C$ by induction hypothesis (since B and C have rank $\leq r$), iff $w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} B \supset C$ by definition 6.9.

(iv) A is $\forall x B$. By definition 2.7, $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, V' \Vdash_{\mathcal{S}} B$ for all existential x-variants V' of V on w. By definition 2.5, V' is an existential x-variant of V on w iff V and V' agree on all predicates, $V'_w(x) \in D_w$ and $V'_w(y) = V_w(y)$ for all $y \neq x$. Take any such V'. By induction hypothesis (since B has rank $\leq r$), $w, V' \Vdash_{\mathcal{S}} B$ iff $w, V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$. So $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$ for all V' such that $V'_w(x) \in D_w$ and $V'_w(y) = V_w(y)$ for all $y \neq x$. In other words, $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, d'_1, \ldots, d'_r \Vdash_{\mathcal{S},I} B$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} \in D_w$ and $d'_i = d_i$ for all $i \neq \rho(x)$, iff $w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \forall x B$ by definition 6.9.

(v) A is $\langle y : x \rangle B$. By definition 2.7, $w, V \Vdash_{\mathcal{S}} \langle y : x \rangle B$ iff $w, V' \Vdash_{\mathcal{S}} B$ where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. By induction hypothesis (since B has rank $\leq r$), $w, V' \Vdash_{\mathcal{S}} B$ iff $w, V'_w(v_1), \ldots, V'_w(v_r) \Vvdash_{\mathcal{S},I} B$. So $w, V \Vdash_{\mathcal{S}} \langle y : x \rangle B$ iff $w, d'_1, \ldots, d'_r \Vvdash_{\mathcal{S},I} B$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} = d_{\rho(y)}$ and $d'_i = d_i$ for all $i \neq \rho(x)$, iff $w, d_1, \ldots, d_r \Vvdash_{\mathcal{S},I} \langle y : x \rangle B$ by definition 6.9.

(vi) A is $\Box B$. By definition 2.7, $w, V \Vdash_{\mathcal{S}} \Box B$ iff $w', V' \Vdash_{\mathcal{S}} B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. By definition 2.6 and totality of $\mathcal{S}$ and V, the latter holds iff V' and V agree on all predicates and for some $C \in K_{w,w'}$ and all variables x, $V_w(x) C V'_{w'}(x)$. Take any such w', V'. By induction hypothesis (since B has rank $\leq r$), $w', V' \Vdash_{\mathcal{S}} B$ iff $w', V'_w(v_1), \ldots, V'_w(v_r) \Vvdash_{\mathcal{S},I} B$. So $w, V \Vdash_{\mathcal{S}} \Box B$ iff $w', V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$, iff $w', d'_1, \ldots, d'_r \Vdash_{\mathcal{S},I} B$ for all $w', d'_1, \ldots, d'_r$ such that $\langle w, d_1, \ldots, d_r \rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n \rangle$ by definition 6.6, iff $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} \Box B$ by definition 6.9. ■

Clearly the evaluation of a formula whose only box operator is $\Box$ does not depend on the accessibility relations associated with the quantificational box operators ($\forall x$, $\langle y : x \rangle$). As we will see, the same is true if we consider the evaluation of modal *schemas* like $\Box A \supset A$, i.e. the set of formulas that result from $\Box p \supset p$ by uniformly substituting arbitrary $\mathcal{L}$-formulas for p. In this way, every purely modal schema corresponds to a constraint on sequential accessibility relations in counterpart structures.

To make this connection between counterpart models and (unimodal) Kripke models even more explicit, we use the following terminology.

Definition 6.11 (Opaque Propositional Guise)
The *n-ary opaque propositional guise of a total counterpart structure* $\mathcal{S} = \langle W, R, U, D, K \rangle$ is the Kripke frame $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle$ where $W^n_{\mathcal{S}}$ is the set of points $\langle w, d_1, \ldots, d_n \rangle$ such that $w \in W, d_1 \in U_w, \ldots, d_n \in U_w$, and $R^n_{\mathcal{S}}$ is the n-accessibility relation for $\mathcal{S}$. The *n-ary opaque propositional guise of a predicate interpretation I* for a language $\mathcal{L}$ on $\mathcal{S}$ is the valuation function V^n on $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle$ such that for every quasi-atomic formula $A \in \mathcal{L}$, $V^n(A) = \{\langle w, d_1, \ldots, d_n \rangle : w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} A\}$.

Lemma 6.12 (Truth-preservation under opaque guises)
For any total counterpart structure $\mathcal{S} = \langle W, R, U, D, K \rangle$, predicate interpretation I on $\mathcal{S}$, world $w \in W$, individuals $d_1, \ldots, d_n \in U_w$, and $\mathcal{L}$-formula A with rank $\leq n$,

$$w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} A \text{ iff } \mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_K A,$$

where $\mathcal{S}^n$ and V^n are the n-ary opaque propositional guises of $\mathcal{S}$ and I respectively.

Proof by induction on A, where quasi-atomic formulas all have complexity zero.

(i) A is quasi-atomic. By definition 6.17, $V^n(A) = \{\langle w, d_1, \ldots, d_n \rangle : w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} A\}$. So $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} A$ iff $\langle w, d_1, \ldots, d_n \rangle \in V^n(A)$, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_K A$ by definition 6.4.

(ii) A is $\neg B$. $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} \neg B$ iff $w, d_1, \ldots, d_n \nVdash_{\mathcal{S},I} B$ by definition 6.9, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \nVdash_K B$ by induction hypothesis, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_K \neg B$ by definition 6.4.

(iii) A is $B \supset C$. $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} B \supset C$ iff $w, d_1, \ldots, d_n \nVdash_{\mathcal{S},I} B$ or $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} C$ by definition 6.9, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \nVdash_K B$ or $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_K C$ by induction hypothesis, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_K B \supset C$ by definition 6.4.

(iv) A is $\Box B$. $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} \Box B$ iff $w', d'_1, \ldots, d'_n \Vvdash_{\mathcal{S},I} B$ for all $w, d'_1, \ldots, d'_r$ such that $\langle w, d_1, \ldots, d_r \rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n \rangle$ by definition 6.9, iff $\mathcal{S}^*, V^*, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_K B$ for all such $w, d'_1, \ldots, d'_r$ by induction hypothesis, iff $\mathcal{S}^*, V^*, \langle w, d_1, \ldots, d_n \rangle \Vdash_K \Box B$ by definition 6.4. ■

Lemma 6.13 (Finite correspondence transfer)
If A is a formula of (unimodal) propositional modal logic that is valid in all and only the Kripke frames in some class F, and $n \in \mathbb{N}$, then the set of $\mathcal{L}$-formulas that result from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas of rank $\leq n$ is positively valid in all and only the total counterpart structures $\mathcal{S} = \langle W, R, U, D, K \rangle$ whose n-ary opaque propositional guise is in F.

Proof Assume A is valid in all and only the Kripke frames in F, and let $p_1, \ldots, p_k$ be the sentence letters in A. Let $\mathcal{S} = \langle W, R, U, D, K \rangle$ be a total counterpart structure whose n-ary opaque propositional guise $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle$ is in F. Suppose for reductio that some formula A' is not (positively) valid in $\mathcal{S}$ that results from A by uniformly substituting the sentence letters p_i in A by $\mathcal{L}$-formulas $p^{\mathcal{L}}_i$ of rank $\leq n$. Then there is an interpretation V on $\mathcal{S}$ and a world $w \in W$ such that $w, V \nVdash_{\mathcal{S}} A'$. By lemma 6.10, this means that $w, d_1, \ldots, d_r \nVdash_{\mathcal{S},I} A'$, where I is V restricted to predicates and $d_1 = V_w(v_1), \ldots, d_r = V_w(v_r)$. By lemma 6.12, it follows that $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \nVdash_K A'$. But then $\mathcal{S}^n, V^{n\prime}, \langle w, d_1, \ldots, d_n \rangle \nVdash_K A$, where $V^{n\prime}$ is such that for all sentence letters p_i in A, $V^{n\prime}(p_i) = V^n(p^{\mathcal{L}}_i)$. This contradicts the assumption that A is valid in $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle$.

We also have to show that the relevant $\mathcal{L}$-formulas are valid *only* in structures $\mathcal{S}$ whose n-ary opaque propositional guise is in F. So let $\mathcal{S}$ be a structure whose guise $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle$ is not in F. Since A is valid only in frames in F, we know that there is some valuation V on $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle$ and some $\langle w, d_1, \ldots, d_n \rangle \in W^n_{\mathcal{S}}$ such that $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w, d_1, \ldots, d_n \rangle \nVdash_K A$. Let A' result from A by uniformly substituting each sentence letter p_i in A by an n-ary predicate P_i followed by the variables $v_1 \ldots v_n$, with distinct predicates for distinct sentence letters. Let I be a predicate interpretation such that for all P_i and $w' \in W$, $I_{w'}(P_i) = \{\langle d'_1, \ldots, d'_n \rangle : \langle w', d'_1, \ldots, d'_n \rangle \in V(p_i)\}$. A simple induction on subformulas B of A shows that for all $\langle w', d'_1, \ldots, d'_n \rangle \in W^n_{\mathcal{S}}$, $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_K B$ iff $w', d'_1, \ldots, d'_n \Vvdash B'$, where B' is B with all p_i replaced by $P_i v_1 \ldots v_n$. Given that $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w, d_1, \ldots, d_n \rangle \nVdash_K A$ it follows that $w, d_1, \ldots, d_n \nVvdash_{\mathcal{S},I} A'$.

(i) B is a sentence letter p_i. Then B' is $P_i v_1 \ldots v_n$. $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_K p_i$ iff $\langle w', d'_1, \ldots, d'_n \rangle \in V(p_i)$ by definition 6.4, iff $\langle d'_1, \ldots, d'_n \rangle \in I_w(P_i)$ by construction of I, iff $w', d'_1, \ldots, d'_n \Vvdash_{\mathcal{S},I} P_i v_1 \ldots v_n$ by definition 6.9.

(ii) B is $\neg C$. Then B' is $\neg C'$, where C' is C with all p_i replaced by $P_i v_1 \ldots v_n$. $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_K \neg C$ iff $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \nVdash_K C$ by definition 6.4, iff $w', d'_1, \ldots, d'_n \nVvdash_{\mathcal{S},I} C'$ by induction hypothesis, iff $w', d'_1, \ldots, d'_n \Vvdash_{\mathcal{S},I} \neg C'$ by definition 6.9.

(iii) B is $C \supset D$. Then B' is $C' \supset D'$, where C' and D' are C and D respectively with all p_i replaced by $P_i v_1 \ldots v_n$. $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_K C \supset D$ iff $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \nVdash_K$

C or $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}}\rangle, V, \langle w', d'_1, \ldots, d'_n\rangle \Vdash_K D$ by definition 6.4, iff $w', d'_1, \ldots, d'_n \nVdash_{\mathcal{S},I} C'$ or $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} D'$ by induction hypothesis, iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} C' \supset D'$ by definition 6.9.

(iv) B is $\Box C$. Then B' is $\Box C'$, where C' is C with all p_i replaced by $P_i v_1 \ldots v_n$. $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}}\rangle, V, \langle w', d'_1, \ldots, d'_n\rangle \Vdash_K \Box C$ iff $\langle W^n_{\mathcal{S}}, R^n_{\mathcal{S}}\rangle, V, \langle w'', d''_1, \ldots, d''_n\rangle \Vdash_K C$ for all $\langle w'', d''_1, \ldots, d''_n\rangle$ with $\langle w', d'_1, \ldots, d'_n\rangle R^n_{\mathcal{S}} \langle w'', d''_1, \ldots, d''_n\rangle$ by definition 6.4, iff $w'', d''_1, \ldots, d''_n \nVdash_{\mathcal{S},I} C'$ for all such $\langle w'', d''_1, \ldots, d''_n\rangle$ by induction hypothesis, iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} \Box C'$ by definition 6.9. ■

Given that a modal schema restricted to the variables $v_1, \ldots, v_n$ defines a constraint on the n-sequential accessibility relation of a counterpart structure $\mathcal{S}$, the unrestricted schema defines a constraint on all sequential accessibility relations. Let's fold these into a single entity.

Definition 6.14 (Sequential accessibility relation)
The *sequential accessibility relation* $R^*_{\mathcal{S}}$ of a total counterpart structure $\mathcal{S}$ is the union of the n-sequential accessibility relations $R^n_{\mathcal{S}}$ of $\mathcal{S}$, i.e. $R^*_{\mathcal{S}} = \bigcup_{n\in\mathbb{N}} R^n_{\mathcal{S}}$.

Definition 6.15 (Opaque Propositional Guise (part 1))
The *opaque propositional guise of a total counterpart structure* $\mathcal{S} = \langle W, R, U, D, K\rangle$ is the disjoint union of the n-ary opaque propositional guises of $\mathcal{S}$, i.e. the Kripke frame $\langle W^*_{\mathcal{S}}, R^*_{\mathcal{S}}\rangle$ such that $R^*_{\mathcal{S}}$ is the sequential accessibility relation of $\mathcal{S}$ and $W^*_{\mathcal{S}}$ is the set of points w^* such that for some $n \in \mathbb{N}$, world $w \in W$ and individuals $d_1, \ldots, d_n \in U_w$, $w^* = \langle w, d_1, \ldots, d_n\rangle$.

Theorem 6.16 ((Positive) correspondence transfer)
If A is a formula of (unimodal) propositional modal logic that is valid in all and only the Kripke frames in some class F, then the set of $\mathcal{L}$-formulas that result from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas is positively valid in all and only the total counterpart structures $\mathcal{S}$ whose opaque propositional guise is in F.

Proof Since validity in propositional modal logic is preserved under disjoint unions, A is valid in the opaque propositional guise of a structure $\mathcal{S}$ iff A is valid in each n-ary opaque propositional guise of $\mathcal{S}$, with $n \in \mathbb{N}$. (See e.g. [Blackburn et al. 2001], p.140, Theorem 3.14.(i).) Thus the opaque propositional guise of $\mathcal{S}$ is in F iff all n-ary opaque propositional guises of $\mathcal{S}$ are in F.

Assume A is valid in all and only the Kripke frames in F. Let A' be an $\mathcal{L}$-formula that results from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas. By lemma 6.13, A' is (positively) valid in all total counterpart structures $\mathcal{S}$ whose n-ary propositional guise is in F, where n is the rank of A'. Any total structure whose propositional guise is in F satisfies this condition.

To show that the A schema is valid *only* in structures $\mathcal{S}$ whose guise is in F, let $\mathcal{S}$ be a structure whose guise is not in F. Then there is some n such that the n-ary guise of $\mathcal{S}$ is not in F. By lemma 6.13, there is an $\mathcal{L}$-substitution instance A' of A with rank n that is not valid in $\mathcal{S}$. ∎

As a union of relations of different arity, R^* is a somewhat gerrymandered entity. It may help to understand statements about R^* as universal statements about its members R^n. For example, the schema $\Box A \supset A$ is valid iff (0) every world can see itself and (1) every individual at every world is its own counterpart (relative to some counterpart relation), (2) every pair of individuals at every world is its own counterpart, and so on. Each clause (i) covers instances of the schema with i free variables.

In Tarski's semantics for first-order logic, every formula is evaluated relative to an infinite sequence of individuals. Accordingly, the points of evaluation in a Tarski-style first-order modal logic would consist of a world together with an infinite sequence of individuals. Quantifiers and modal operators shift these points of evaluation. Thus one might expect that modal schemas constrain the infinitary accessibility relation R^∞ that holds between a pair $\langle w, s\rangle$ of a world w and an infinite sequence s on U_w and another such pair $\langle w', s'\rangle$ iff for some $C \in K_{w,w'}$, each element s_i of s has the corresponding element s'_i of s' as C-counterpart. But this is false. For example, $\Box\bot$ is valid in a Kripke frame iff no point can see itself. Suppose in some structure $\mathcal{S}$, no point $\langle w, s\rangle$ stands in R^∞ to any point $\langle w', s'\rangle$, although for some w, w', wRw'. Since $\neg\langle w, s\rangle R^\infty \langle w', s'\rangle$, we know that there is no $C \in K_{w,w'}$ such that $s_1 C s'_1, s_2 C s'_2, \ldots$. In other words, for all $C \in K_{w,w'}$ there is some i such that $\neg s_i C s'_i$. But (as long as $K_{w,w'}$ is infinite) this is compatible with the fact that for all i there is a $C \in K_{w,w'}$ such that $s_1 C s'_1, \ldots, s_i C s'_i$.[6] The general point is that in a Kripke model with points of the form $\langle w, s\rangle$, formulas can express arbitrary properties of world and infinite sequences of individuals; by contrast, formulas of $\mathcal{L}$ can only talk about a finite initial segment of a sequences.

Thus far, I have set aside negative counterpart models. In negative models, variables can be undefined, so sequential accessibility relations must be redefined to hold between pairs of a world w and a possibly *gappy* sequence of individuals from D_w, i.e. a partial function from numbers to members of D_w. Now we could re-run the above arguments, but we can also cut all this short by using lemma 2.12.

6 Thanks here to Joel David Hamkins and Sam Roberts.

Definition 6.17 (Opaque Propositional Guise (part 2))
The *opaque propositional guise of a partial counterpart structure* $\mathcal{S}$ is the opaque propositional guise of its positive transpose $\mathcal{S}^+$. Accordingly, the *sequential accessibility relation* $R^*_{\mathcal{S}}$ *of* $\mathcal{S}$ is the sequential accessibility relation of $\mathcal{S}^+$.

Corollary 6.18 (Negative correspondence transfer)
If A is a formula of (unimodal) propositional modal logic that is valid in all and only the Kripke frames in some class F, then the set of $\mathcal{L}$-formulas that result from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas is negatively valid in all and only the single-domain counterpart structures $\mathcal{S}$ whose opaque propositional guise is in F.

Proof Assume A is valid in all and only the Kripke frames in F. Let A' be an $\mathcal{L}$-formula that results from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas. Let $\mathcal{S}$ be a single-domain counterpart structure whose guise is in F, and let w, V be a world from $\mathcal{S}$ and a partial interpretation on $\mathcal{S}$. By lemma 2.12, $w, V \Vdash_{\mathcal{S}} A'$ iff $w, V^+ \Vdash_{\mathcal{S}^+} A'$. By theorem 6.16, $w, V^+ \Vdash_{\mathcal{S}^+} A'$. So A' is negatively valid in $\mathcal{S}$.

Now let $\mathcal{S}$ be a structure whose guise is not in F. By theorem 6.16, there is a substitution instance A' of A, a world w and a total interpretation V' on $\mathcal{S}^+$ such that $w, V' \nVdash_{\mathcal{S}^+} A'$. Define V so that V and V' agree on all predicates and for all worlds w' and variables x, $V_{w'}(x)$ is $V'_{w'}(x)$ if $V'_{w'}(x) \in D_w$, otherwise $V_{w'}(x)$ is undefined. Then V' is the positive transpose of V. By lemma 2.12, $w, V \Vdash_{\mathcal{S}} A'$ iff $w, V' \Vdash_{\mathcal{S}^+} A'$. So $w, V \nVdash_{\mathcal{S}} A'$. So A' is not negatively valid in $\mathcal{S}$. ■

Here are some applications of theorems 6.16 and 6.18.

Corollary 6.19 (Correspondence facts)

1. The schema $\Box A \supset A$ is valid in a counterpart structure $\mathcal{S}$ iff $R^*_{\mathcal{S}}$ is reflexive.
2. The schema $A \supset \Box\Diamond A$ is valid in a counterpart structure $\mathcal{S}$ iff $R^*_{\mathcal{S}}$ is symmetrical.
3. The schema $\Box A \supset \Box\Box A$ is valid in a counterpart structure $\mathcal{S}$ iff $R^*_{\mathcal{S}}$ is transitive.
4. The schema $\Diamond A \supset \Box\Diamond A$ is valid in a counterpart structure $\mathcal{S}$ iff $R^*_{\mathcal{S}}$ is euclidean.

There are of course further aspects of structures that can only be captured in quantified modal logic.

Definition 6.20 (Types of structures)
A structure $\mathcal{S} = \langle W, R, U, D, K \rangle$ is

total if every individual at every world has at least one counterpart at every accessible world: whenever wRw' and $d \in U_w$ then there is a $d' \in U_{w'}$ with $\langle d, w \rangle C \langle d', w' \rangle$;

functional if every individual at every world has at most one counterpart at every accessible world;

inversely functional if no two individuals at any world have a common counterpart at some accessible world;

injective if it is both functional and inversely functional.

[To be continued…]

7 Canonical models

We will use the canonical model technique for proving strong completeness. Let me briefly review the basic idea.

Let's hold fixed a particular language $\mathcal{L}$, with or without substitution. A formula A of $\mathcal{L}$ is a *syntactic consequence in a logic* L of a set Γ of $\mathcal{L}$-formulas L, for short: $\Gamma \vdash_L A$, iff there are 0 or more sentences $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_L B_1 \wedge \ldots \wedge B_n \supset A$. (For $n = 0$, $B_1 \wedge \ldots \wedge B_n \supset A$ is A.) Γ is *L-consistent* iff there are no members $A_1, \ldots, A_n$ of Γ such that $\vdash_L \neg(A_1 \wedge \ldots \wedge A_n)$.

A logic L is *weakly complete* with respect to a class of models $\mathbb{M}$ iff L contains every formula valid in $\mathbb{M}$: whenever A is valid in $\mathbb{M}$, then $\vdash_L A$. Equivalently, every formula $A \notin L$ is false at some world in some model in $\mathbb{M}$. L is *strongly complete* with respect to $\mathbb{M}$ iff whenever A is a semantic consequence of a set of formulas Γ in $\mathbb{M}$, then $\Gamma \vdash_L A$. Since $\Gamma \nvdash_L A$ iff $\Gamma \cup \{\neg A\}$ is L-consistent, and A is a semantic consequence of Γ in $\mathbb{M}$ iff no world in any model in $\mathbb{M}$ verifies all members of $\Gamma \cup \{\neg A\}$, this means that L is strongly complete with respect to $\mathbb{M}$ iff for every L-consistent set of formulas Γ there is a world in some model in $\mathbb{M}$ at which all members of Γ are true.

To prove strong completeness, we associate with each logic L a *canonical model* $\mathcal{M}_L$. The worlds of $\mathcal{M}_L$ are construed as maximal L-consistent sets of formulas, and it is shown that a formula A is true at a world in $\mathcal{M}_L$ iff A is a member of that world. Since every L-consistent set of formulas can be extended to a maximal L-consistent set, it follows that every L-consistent set of formulas is verified at some world in $\mathcal{M}_L$, and therefore that L is strongly complete with respect to every model class that contains $\mathcal{M}_L$.

To ensure that what's true at a world are precisely the formulas it contains, the interpretation V in a canonical model assigns to each variable x at each world w some individual $[x]_w$ and to each predicate P at w the set of n-tuples $\langle [x_1]_w, \ldots, [x_n] \rangle$ such that $Px_1 \ldots x_n \in w$. The customary way to make this work is to identify $[x]_w$ with the class of variables z such that $x = z \in w$. The domains therefore consist of equivalence classes of variables.

A well-known problem now arises from the fact that first-order logic does not require every individual to have a name. This means that there are consistent sets Γ that contain $\exists xFx$ as well as $\neg Fx_i$ for every variable x_i. If we extend Γ to a maximal consistent set w and apply the construction just outlined, then $V_w(F)$ would be the empty set. So we would have $w, V \Vdash \neg\exists xFx$, although $\exists xFx \in w$. To avoid this, one requires that the worlds in a canonical model are all *witnessed* so that whenever an existential formula $\exists xFx$ is in w, then some witnessing instance Fy is in w as well. But we still want the set Γ to be verified at some world. So the worlds are construed in a larger language $\mathcal{L}^*$ that adds infinitely many new variables to the original language $\mathcal{L}$. The new variables may then serve as witnesses. (In the new language, there are again consistent sets of sentences that are not included in any world, but not so in the old language.)

In modal logic, this problem reappears in another form. Assume Γ contains $\Diamond\exists xFx$ but also $\Box\neg Fx_i$ for every $\mathcal{L}^*$-variable x_i. Using Kripke semantic, we then need a world w' accessible from the Γ-world that verifies all instances of $\neg Fx_i$, as well as $\exists xFx$. But then w' isn't witnessed!

One way out is to stipulate that worlds in $\mathcal{M}_L$ must be *modally witnessed* in the sense that whenever $\Diamond\exists xA \in w$, then $\Diamond[y/x]A \in w$ for some (possibly new) variable y. Metaphysically speaking, this means that whenever it is possible that something is so-and-so, then we can point at some object at the actual world which is possibly so-and-so. In single-domain models, this has the unfortunate consequence of rendering the Barcan Formula valid. In dual-domain semantics, the "modal witness" can come from the outer domain, so that $\forall x\Box Fx$ does not entail $\Box\forall xFx$. However, if the relevant logic is classical rather than free, the Barcan Formula still comes out valid, despite the fact that it is not entailed by the principles of classical first-order logic and K.

Counterpart semantics provides a less drastic response that does not introduce undesired validities and thereby reduce the applicability of the canonical model technique. In counterpart semantics, the truth of $\Box\neg Fx_i$ at some world w only requires that $\neg Fx_i$ is true at w' under all *w'-images* V' of V at w – i.e. under interpretations V' such that $V'_{w'}(x_i)$ is some counterpart of $V_w(x_i) = [x_1]_w$. Suppose, for example, that $[x_1]_w = \{x_1\}$

and each individual $[x_i]_w$ at w has $[x_{i+1}]_{w'}$ as unique counterpart at w'. Then the truth of $\Box\neg Fx_1, \Box\neg Fx_2$, etc. at w only requires that $\neg Fx_1, \neg Fx_2$, etc. are true at w' under an assignment of $[x_2]_{w'}$ to x_1, $[x_3]_{w'}$ to x_2, etc. So $Fx_2, Fx_3, \ldots \in w'$, but the variable x_1 becomes available to serve as a witness for $\Diamond\exists xFx$.

Here we exploit the fact that in counterpart semantics truth at a world "considered as actual" can come apart from truth at a world "considered as counterfactual". In the canonical model, membership in a world only coincides with truth at the world "as actual": $A \in w$ iff $w, V \Vdash A$. If w' contains Fx_1, it follows that $w', V \Vdash Fx_1$. On the other hand, when we look at w' ("as counterfactual") from the perspective of w, we evaluate formulas not by the original interpretation function V, but by an image V' of V. Given that $V'_{w'}(x_1) = [x_2]_{w'}$ and $Fx_2 \notin w'$, $w', V' \nVdash Fx_1$.

Unfortunately, this creates a complication. Suppose we want to show that for every formula A and world w in $\mathcal{M}_L$,

(8) $\Box A \in w$ iff $w, V \Vdash \Box A$,

where V is the interpretation function of $\mathcal{M}_L$. Proceeding by induction on complexity of A, we can assume that for all w,

(9) $A \in w$ iff $w, V \Vdash A$.

In standard Kripke semantics, we now only need to stipulate that w' is accessible from w iff w' contains all A for which w contains $\Box A$. This means that $\Box A \in w$ iff $A \in w'$ for all w-accessible w'; by (9), the latter holds iff $w', V \Vdash A$ for all such w', i.e. iff $w, V \Vdash \Box A$ by the semantics of the box. In counterpart semantics, this line of thought no longer goes through, since $w, V \Vdash \Box A$ only means that A is true at all accessible worlds w' *considered as counterfactual*: $w', V' \Vdash A$. By contrast, (9) only considers worlds *as actual*; it does not tell us that $A \in w$ iff $w, V' \Vdash A$.

Take a concrete example. Suppose w and w' look as follows.

$$w : \{x \neq y, \Box x \neq y, \Box Fx, \Box Fy, \ldots\}$$
$$w' : \{\neg Fx, Fu, Fv, u \neq y, \ldots\}$$

We can tell that $[x]_{w'}$ does not qualify as counterpart of $[x]_w$, since it doesn't satisfy the "modal profile" w attributes to $[x]_w$: w contains $\Box Fx$, so all counterparts of $[x]_w$ should satisfy F. Both $[u]_{w'}$ and $[v]_{w'}$ meet this condition. So we might say that both of them are counterparts of $[x]_w$. But then they should also be counterparts of $[y]_w$, and we get a violation of the "joint modal profile" expressed by $\Box x \neq y$, which requires that no counterpart of $[x]_w$ is identical to any counterpart of $[y]_w$. Structurally, this is the "problem of internal relations" noted in [Hazen 1979]. In response, we assume that there can be multiple counterpart relations linking the individuals at w to those at w'. One relation links $[x]_w$ with $[u]_{w'}$ and $[y]_w$ with $[v]_w$, another $[x]_w$ with $[v]_{w'}$ and $[y]_w$ with $[u]_{w'}$. So $[x]_w$ does have both $[u]_{w'}$ and $[v]_{w'}$ as counterpart, but the pair $\langle [x]_w, [y]_w \rangle$ has only two rather than four counterparts.

It proves convenient to impose a further restriction on canonical counterpart relations: every counterpart relation C in a canonical model corresponds to a transformation τ so that $[x]_w$ at w has $[x^\tau]_{w'}$ at w' as counterpart (unless $[x^\tau]_{w'}$ is empty – see below). This means that if an individual $[x]_w$ at w has two counterparts at w' under the same counterpart relation, then there must be at least two variables x, y in $[x]_w$, so that $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$ can serve as the two counterparts.

So here is how we might construct the accessibility and counterpart relations. Let's say that w' is *accessibility from w' via* a transformation τ, for short: $w \xrightarrow{\tau} w'$, iff w' contains A^τ whenever w contains $\Box A$. Define wRw' to be true iff $w \xrightarrow{\tau} w'$ for some τ, and let C be a counterpart relation between w and w' iff $C = \{\langle [x]_w, [y]_{w'} \rangle : x^\tau = y\}$ for some $w \xrightarrow{\tau} w'$.

If $w \xrightarrow{\tau} w'$, then $[x]_w$ has $[x^\tau]_{w'}$ as counterpart. Since $[x^\tau]_{w'} = V_{w'}(x^\tau)$ and $V_{w'}(x^\tau) = V^\tau_{w'}(x)$, V^τ is a w'-image of V at w. If V^τ were the only w'-image of V at w, it would be easy to prove that $\Box A \in w$ iff $w, V \Vdash \Box A$. Assume $\Box A \in w$. Then $A^\tau \in w'$ whenever $w \xrightarrow{\tau} w'$. The induction hypothesis (9) tells us that $A^\tau \in w'$ iff $w', V \Vdash A^\tau$. By the transformation lemma (lemma 3.13), $w', V^\tau \Vdash A$ iff $w', V \Vdash A^\tau$: A is true at w' as counterfactual iff A^τ is true at w' as actual. So $\Box A \in w$ iff for all accessible w', there is a transformation τ such that $w', V^\tau \Vdash A$. If V^τ is the only w'-image of V_w at w', it follows that $\Box A \in w$ iff $w, V \Vdash \Box A$.

The remaining problem is that V^τ may not be the only w'-image of V at w – and not only because there can be several τ with $w \xrightarrow{\tau} w'$. For example, assume w contains $x = y$ but not $\Box x = y$. Then there is some world w' and transformation τ such that w' contains $x^\tau \neq y^\tau$. I.e., the individual $[x]_w = [y]_w = \{x, y, \ldots\}$ at w has two τ-induced counterparts at w', $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$, which V^τ assigns (at w') to x and y, respectively. But then there will also be another w'-image of V at w which assigns, for example, $[y^\tau]_{w'}$ to both x and y.

Here is a case where this leads to trouble. Assume we are working with a positive logic without explicit substitution. Again let w contain $x = y$ but not $\Box x = y$, so that for some $w \xrightarrow{\tau} w'$, w' contains $x^\tau \neq y^\tau$. Assume further that w contains $\Box\Diamond x \neq y$. So w' contains $\Diamond x^\tau \neq y^\tau$. To verify $\Box\Diamond x \neq y$ at w, we need to ensure that $w', V' \Vdash \Diamond x \neq y$ for all w'-images V' of V at w, not just for V^τ. Consider the image V' that assigns $[y^\tau]_{w'}$ to both x and y. To ensure that $w', V' \Vdash \Diamond x \neq y$, there must be some $w' \xrightarrow{\sigma} w''$ and $V'_{w'} \triangleright V''_{w''}$ with $w'', V'' \Vdash x \neq y$. $V'_{w'} \triangleright V''_{w''}$ means that there is a σ such that $w' \xrightarrow{\sigma} w''$ and $V'_{w'}(x) C^\sigma V''_{w''}(x)$ for all x, where C^σ is the counterpart relation induced by σ. In other words, we need a transformation σ, world w'' and interpretation V'' such that $w'', V'' \Vdash x \neq y$, where $w' \xrightarrow{\sigma} w''$ and V'' is such that for all x there is a $z \in V'_{w'}(x)$ with

$z^\sigma \in V''_{w''}(x)$. Since $V'_{w'}(x) = V'_{w'}(y) = [y^\tau]_{w'}$, this means that $[y^\tau]_{w'}$ must have two counterparts at some w'' relative to the same transformation σ. But so far, we have no guarantee that this is the case. There has to be a variable z other than y^τ such that w' contains $z = y^\tau$ as well as $\Diamond z \neq y^\tau$. The latter ensures that $z^\sigma \neq (y^\tau)^\sigma \in w''$ for some $w \xrightarrow{\sigma} w''$; $[z^\sigma]_{w''}$ and $[(y^\tau)^\sigma]_{w''}$ are then both counterparts at w'' of $[y^\tau]_{w'}$. Hence we complicate the definition of $w \xrightarrow{\tau} w'$. We stipulate that if w' does not contain $z = y^\tau$ and $\Diamond z \neq y^\tau$ for some suitable z, then w' is not τ-accessible from w. In general, if w contains $\Box A$ as well as $x = y$, and x is free in A, then for w' to be accessible from w via τ, we require that it must contain not only A^τ, but also $z = y^\tau$ and $[z/x^\tau]A^\tau$, for some z not free in A^τ.

This requirement might be easier to understand if we consider the same situation in a language with substitution. Here $\Box\Diamond x \neq y$ and $x = y$ entail $\Box\langle y : x\rangle\Diamond x \neq y$ (by ($\mathrm{LL_s}$) and ($\mathrm{S}\Box$)). By the original, simple definition of $w \xrightarrow{\tau} w'$, each world w' accessible from w via τ must contain $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$. This formula says that $[y^\tau]_{w'}$ has multiple counterparts at some accessible world w''. Before we worry about images other than V^τ, we ought to make sure that $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$ is true at w' under V^τ. This requires that there is a variable z other than y^τ such that w' contains $z = y^\tau$ and $\Diamond z \neq y^\tau$. In effect, z is a kind of witness for the substitution formula $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$. Just as an existential formula $\exists x A$ must be witnessed by an instance $[z/x]A$, a substitution formula $\langle y : x\rangle A$ must be witnessed by $[z/x]A$ together with $z = y$. Loosely speaking, $\langle y : x\rangle A(x)$ says that y is identical to some x such that $A(x)$. In a canonical model, we want a concrete witness z so that y is identical to z and $A(z)$. y itself may not serve that purpose, because $\langle y : x\rangle A(x)$ does not guarantee $A(y)$.

This requirement of substitutional witnessing entails that if w contains $\Box A$, then any τ-accessible w' contains not only A^τ, but also $z = y^\tau$ and $[z/x^\tau]A^\tau$ (for some suitable z). So we don't need to complicate the accessibility relation. In our example, since w' contains A^τ whenever w contains $\Box A$, w' contains $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$, which settles that $[y^\tau]_{w'}$ has two counterparts at some accessible world. Without substitution, $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$ is inexpressible (see lemma 3.10). So we have to limit the accessible worlds by requiring membership of the relevant witnessing formulas in addition to A^τ.

On to the details. Let $\mathcal{L}$ be some language with or without substitution and L a positive or strongly negative quantified modal logic in $\mathcal{L}$. Define the extended language $\mathcal{L}^*$ by adding infinitely many new variables Var^+ to $\mathcal{L}$.

> Definition 7.1 (Henkin set)
> A *Henkin set* for L is a set H of $\mathcal{L}^*$-formulas that is
>
> 1. *L-consistent*: there are no $A_1, \ldots, A_n \in H$ with $\vdash_{L(\mathcal{L}^*)} \neg(A_1 \land \ldots \land A_n)$,
> 2. *maximal*: for every $\mathcal{L}^*$-formula A, H contains either A or $\neg A$,

3. *witnessed*: whenever H contains an existential formula $\exists xA$, then there is a variable $y \notin \mathit{Var}(A)$ such that H contains $[y/x]A$ as well as Ey, and

4. *substitutionally witnessed*: whenever H contains a substitution formula $\langle y : x\rangle A$ as well as $y = y$, then there is a variable $z \notin \mathit{Var}(\langle y : x\rangle A)$ such that H contains $y = z$.

I write $\mathcal{H}_L$ for the class of Henkin sets for L in $\mathcal{L}^*$.

If L is without substitution, the fourth clause is trivial.

Above I said that witnessing a substitution formula $\langle y : x\rangle A$ requires $y = z$ as well as $[z/x]A$, but in fact $y = z$ is enough, since $[z/x]A$ follows from $\langle y : x\rangle A$ and $y = z$ by (LV2) (lemma 5.13). I have also added the condition that H contains $y = y$. In negative logics, a Henkin set may contain $y \neq y$ as well as $\langle y : x\rangle A$; adding $y = z$ would render the set inconsistent.

The requirement of substitutional witnessing generalises to substitution sequences: if H contains a substitution formula $\langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$ as well as $y_i = y_i$ for all y_i in $y_1, \ldots, y_n$, then there are (distinct) new variables $z_1, \ldots, z_n$ such that H contains $y_1 = z_1, \ldots, y_n = z_n$ as well as $[z_1, \ldots, z_n/x_1, \ldots, x_n]A$. This is easily proved by induction on n. Suppose H contains $\langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$. By definition 3.14, this is $\langle y_n : v\rangle\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle v : x_n\rangle A$, where v is new. Witnessing requires $y_n = z_n \in H$ and (hence) $[z_n/v]\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle v : x_n\rangle A = \langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z_n : x_n\rangle A \in H$ for some new z_n. By induction hypothesis, the latter means that there are (distinct) $z_1, \ldots, z_{n-1} \notin \mathit{Var}(\langle z_n : x_n\rangle A)$ such that H contains $y_1 = z_1, \ldots, y_{n-1} = z_{n-1}$ as well as $[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]\langle z_n : x_n\rangle A$. Since all the x_i and z_i are pairwise distinct, $[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]\langle z_n : x_n\rangle A$ is $\langle z_n : x_n\rangle[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]A$. By (SC1), it follows that $[z_n/x_n][z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]A = [z_1, \ldots, z_n/x_1, \ldots, x_n]A \in H$.

Definition 7.2 (Variable classes)
For any Henkin set H, define $\sim_H$ to be the binary relation on the variables of $\mathcal{L}^*$ such that $x \sim_H y$ iff $x = y \in H$. For any variable x, let $[x]_H$ be $\{y : x \sim_H y\}$.

Lemma 7.3 ($\sim$-Lemma)
$\sim_H$ is transitive and symmetrical.

Proof Immediate from lemmas 4.10 and 5.12. ■

Definition 7.4 (Accessibility via transformations)
Let w, w' be Henkin sets and τ a transformation.

If $\mathcal{L}$ is with substitution, then w' *is accessible from* w *via* τ, for short: $w \xrightarrow{\tau} w'$, iff for every $\mathcal{L}$-formula A, if $\Box A \in w$, then $A^\tau \in w'$.

If $\mathcal{L}$ is without substitution, then $w \xrightarrow{\tau} w'$ iff for every $\mathcal{L}$-formula A and variables $x_1 \ldots x_n$, $y_1, \ldots, y_n$ ($n \geq 0$) such that the $x_1 \ldots x_n$ are pairwise distinct members of $\mathit{Varf}(A)$, if $x_1 = y_1 \land \ldots \land x_n = y_n \land \Box A \in w$ and $y_1^\tau = y_1^\tau \land \ldots \land y_n^\tau = y_n^\tau \in w'$, then there are variables $z_1 \ldots z_n \notin \mathit{Var}(A^\tau)$ such that $z_1 = y_1^\tau \land \ldots \land z_n = y_n^\tau \land [z_1 \ldots z_n / x_1^\tau \ldots x_n^\tau] A^\tau \in w'$.

This generalises the witnessing requirements on accessible world as explained above to multiple variables and negative logics. (In this case, the generalised version for n variable pairs is not entailed by the requirement for a single pair, unlike in the case of substitutional witnessing.) Note that the $x_1, \ldots, x_n$ need not be *all* the free variables in A. Also recall from p.7 that a conjunction of zero sentences is the tautology $\top$; so for $n = 0$, the accessibility requirement says that if $\top \land \Box A \in w$ and $\top \in w'$, then $\top \land A^\tau \in w'$ – equivalently: if $\Box A \in w$, then $A^\tau \in w'$.

Definition 7.5 (Canonical model)
The *canonical model* $\langle W, R, U, D, K, V \rangle$ for L is defined as follows.

1. The *worlds* W are the Henkin sets $\mathcal{H}_L$.
2. For each $w \in W$, the *outer domain* U_w comprises the non-empty sets $[x]_w$, where x is a $\mathcal{L}^*$-variable.
3. For each $w \in W$, the *inner domain* D_w comprises the sets $[x]_w$ for which $Ex \in w$.
4. The *accessibility relation* R holds between world w and world w' iff there is some transformation τ such that $w \xrightarrow{\tau} w'$.
5. C is a *counterpart relation* $\in K_{w,w'}$ iff there is a transformation τ such that (i) $w \xrightarrow{\tau} w'$ and (ii) for all $d \in U_w, d' \in U_{w'}$, dCd' iff there is an $x \in d$ such that $x^\tau \in d'$.
6. The *interpretation* V is such that for all $\mathcal{L}^*$-variables x, $V_w(x)$ is either $[x]_w$ or undefined if $[x]_w = \emptyset$, and for all non-logical predicates P, $V_w(P) = \{\langle [x_1]_w, \ldots, [x_n]_w \rangle : Px_1 \ldots x_n \in w\}$.

Clause 6 takes into account the fact that in negative logics, $\neg Ex$ entails $x \neq y$ for every variable y. So if $\neg Ex \in w$, then $[x]_w$ is the empty set. However, we don't want to say that empty terms denote the empty set (so that $\emptyset \in D_w$, and $x = x$ would have to be true). Instead, the canonical interpretation assigns to each variable x at w the set $[x]_w$, *unless that set is empty*, in which case $V_w(x)$ remains undefined. Similarly, clause 5 ensures that $[x]_w$ at w has $[x^\tau]_{w'}$ as counterpart at w' only if $[x^\tau]_{w'} \neq \emptyset$.

The term '$\{\langle [x_1]_w, \ldots, [x_n]_w \rangle : Px_1 \ldots x_n \in w\}$' in clause 4 is meant to denote the set of n-tuples $\langle d_1, \ldots, d_n \rangle$ for which there are variables $x_1, \ldots, x_n$ such that $d_1 = [x_1]_w$ and ... and $d_n = [x_n]_w$ and $Px_1 \ldots x_n \in w$. These d_i are guaranteed to be non-empty because $x_i = x_i \in w$ whenever $Px_1 \ldots x_n \in w$: if L is positive, then $\vdash_L z_i = z_i$ by (=R); if L is negative, then $\vdash_L Pz_1 \ldots z_n \supset Ez_i$ by (Neg) and hence $\vdash_L Pz_1 \ldots z_n \supset z_i = z_i$ by ($\forall$=R) and (FUI*).

Lemma 7.6 (Charge of canonical models)
If L is positive, then the canonical model for L is positive. If L is strongly negative, then the canonical model for L is negative.

Proof If L is positive, then for all $\mathcal{L}^*$-variables x, every Henkin set for L contains $x = x$ (by (=R)). So $V_w(x) = [x]_w$ is never empty. Nor is $[x^\tau]_{w'}$, for any world w' with $w \xrightarrow{\tau} w'$. So everything at any world has a counterpart at every accessible world under every counterpart relation. So the canonical model for a positive logic is positive.

If L is strongly negative, then every Henkin set for L contains $x = x \supset Ex$, for all $\mathcal{L}^*$-variables x (by (Neg)). So $V_w(x) = [x]_w \neq \emptyset$ iff $Ex \in w$, which means that $D_w = U_w$ for all worlds w in the model. So the canonical model for a strongly negative logic is negative. ■

Lemma 7.7 (Extensibility Lemma)
If Γ is an L-consistent set of $\mathcal{L}^*$-sentences in which infinitely many $\mathcal{L}^*$-variables do not occur, then there is a Henkin set $H \in \mathcal{H}_L$ such that $\Gamma \subseteq H$.

Proof Let $S_1, S_2, \ldots$ be an enumeration of all $\mathcal{L}^*$-sentences, and $z_1, z_2, \ldots$ an enumeration of the unused $\mathcal{L}^*$-variables in such a way that $z_i \notin Var(S_1 \wedge \ldots \wedge S_i)$. Let $\Gamma_0 = \Gamma$, and define Γ_n for $n \geq 1$ as follows.

(i) If $\Gamma_{n-1} \cup \{S_n\}$ is not L-consistent, then $\Gamma_n = \Gamma_{n-1}$;

(ii) else if S_n is an existential formula $\exists x A$, then $\Gamma_n = \Gamma_{n-1} \cup \{\exists x A, [z_n/x]A, Ez_n\}$;

(iii) else if S_n is a substitution formula $\langle y : x \rangle A$, then $\Gamma_n = \Gamma_{n-1} \cup \{\langle y : x \rangle A, y = y \supset y = z_n\}$;

(iv) else $\Gamma_n = \Gamma_{n-1} \cup \{S_n\}$.

Define w as the union of all Γ_n. We show that w is a Henkin set for L.

1. w is L-consistent. This is shown by proving that Γ_0 is L-consistent and that whenever Γ_{n-1} is L-consistent, then so is Γ_n. It follows that no finite subset of w is L-inconsistent, and hence that w itself is L-consistent. The base step, that Γ_0 is L-consistent is given by assumption. Now assume (for $n > 0$) that Γ_{n-1} is L-consistent. Then Γ_n is constructed by applying one of (i)–(iv).

   a) If case (i) in the construction applies, then $\Gamma_n = \Gamma_{n-1}$, and so Γ_n is also L-consistent.

   b) Assume case (ii) in the construction applies, and suppose that $\Gamma_n = \Gamma_{n-1} \cup \{\exists x A, [z_n/x]A, Ez\}$ is L-inconsistent. Then there is a finite subset $\{C_1, \ldots, C_m\} \subseteq \Gamma_{n-1}$ such that

      1. $\vdash_L \neg(C_1 \wedge \ldots \wedge C_m \wedge \exists x A \wedge [z_n/x]A \wedge E z_n)$.

      Let $\underline{C}$ abbreviate $C_1 \wedge \ldots \wedge C_m$. Then

2.	$\vdash_L \underline{C} \wedge \exists x A \supset (E z_n \supset \neg[z_n/x]A)$	(1)
3.	$\vdash_L \forall z_n(\underline{C} \wedge \exists x A) \supset \forall z_n E z_n \supset \forall z_n \neg[z_n/x]A$	(2, (UG), (UD))
4.	$\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n(\underline{C} \wedge \exists x A)$	((VQ), z_n not in Γ_{n-1})
5.	$\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n E z_n \supset \forall z_n \neg[z_n/x]A.$	(3, 4)
6.	$\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n \neg[z_n/x]A.$	(5, ($\forall$Ex))
7.	$\vdash_L \forall z_n \neg[z_n/x]A \leftrightarrow \forall x \neg A$	((AC), $z_n \notin \mathit{Var}(A)$)
8.	$\vdash_L \underline{C} \wedge \exists x A \supset \neg\exists x A.$	(6, 7)

      So $\{C_1, \ldots C_m, \exists x A\}$ is not L-consistent, contradicting the assumption that clause (ii) applies.

   c) Assume case (iii) in the construction applies (hence L is with substitution), and suppose that $\Gamma_n = \Gamma_{n-1} \cup \{\langle y : x\rangle A, y{=}y \supset y{=}z_n\}$ is L-inconsistent. Then there is a finite subset $\{C_1, \ldots, C_m\} \subseteq \Gamma_{n-1}$ such that

      1. $\vdash_L \neg(\underline{C} \wedge \langle y : x\rangle A \wedge (y{=}y \supset y{\neq}z))$.

      (As before, $\underline{C}$ is $C_1 \wedge \ldots \wedge C_m$.) But then

2.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset y{=}y \wedge y{\neq}z_n$	(1)
3.	$\vdash_L \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A \supset y{=}y \wedge y{\neq}z_n)$	(2, ($\mathrm{Sub_s}$))
4.	$\vdash_L \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A) \supset \langle y : z_n\rangle y{=}y \wedge \langle y : z_n\rangle y{\neq}z_n$	(3, (S$\supset$), (S$\neg$))
5.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A)$	((VS), z_n not in Γ_{n-1}, S_n)
6.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset \langle y : z_n\rangle y{=}y \wedge \langle y : z_n\rangle y{\neq}z_n$	(4, 5)
7.	$\vdash_L \langle y : z_n\rangle y{\neq}z_n \leftrightarrow y{\neq}y$	(SAt)
8.	$\vdash_L \langle y : z_n\rangle y{=}y \leftrightarrow y{=}y$	(SAt)
9.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset (y{=}y \wedge y{\neq}y).$	(6, 7, 8)

So $\{C_1, \ldots, C_m, \langle y : x\rangle A\}$ is L-inconsistent, contradicting the assumption that clause (iii) applies.

d) Assume case (iv) in the construction applies. Then $\Gamma_n = \Gamma_{n-1} \cup \{S_n\}$ is L-consistent, since otherwise (i) would have applied.

2. w is maximal. Assume some formula S_n is not in w. Then case (i) applied to S_n, so $\Gamma_{n-1} \cup \{S_n\}$ is not L-consistent. So there are $C_1, \ldots, C_m \in \Gamma_{n-1}$ such that $\vdash_L C_1 \wedge \ldots C_m \supset \neg S_n$. Similarly, if $S_k = \neg S_n$ is not in w, then there are $D_1, \ldots, D_l \in \Gamma_{k-1}$ such that $\vdash_L D_1 \wedge \ldots D_l \supset \neg S_k$. By (PC), it follows that there are $C_1, \ldots, C_m, D_1, \ldots D_l \in w$ such that

$$\vdash_L C_1 \wedge \ldots \wedge C_m \wedge D_1 \wedge \ldots \wedge D_l \supset (\neg S_n \wedge \neg\neg S_n).$$

But then w is inconsistent, contradicting what was just shown under 1.

3. w is witnessed. This is guaranteed by clause (ii) of the construction and the fact that the $z_n \notin \mathit{Var}(S_n)$. ■

4. w is substitutionally witnessed. This is guaranteed by clause (iii) and the fact that the $z_n \notin \mathit{Var}(S_n)$. ■

Lemma 7.8 (Existence Lemma)
If w is a world in the canonical model for L, A a formula with $\Diamond A \in w$, and τ any transformation whose range excludes infinitely many variables of $\mathcal{L}$, then there is a world w' in the model such that $w \xrightarrow{\tau} w'$ and $A^\tau \in w'$.

Proof I first prove the lemma for logics L with substitution. Let $\Gamma = \{A^\tau\} \cup \{B^\tau : \Box B \in w\}$. Suppose Γ is not L-consistent. Then there are $B_1^\tau, \ldots, B_n^\tau$ with $\Box B_i \in w$ such that $\vdash_L B_1^\tau \wedge \ldots \wedge B_n^\tau \supset \neg A^\tau$. By definition 3.3, this means that $\vdash_L (B_1 \wedge \ldots B_n \supset \neg A)^\tau$, and so $\vdash_L B_1 \wedge \ldots \wedge B_n \supset \neg A$ by (Sub$^\tau$). By (Nec) and (K), $\vdash_L \Box B_1 \wedge \ldots \wedge \Box B_n \supset \Box\neg A$. But then w contains both $\Diamond A$ and $\neg\Diamond A$, which is impossible because w is L-consistent. So Γ is L-consistent.

Since the range of τ excludes infinitely many variables, by the extensibility lemma, $\Gamma \subseteq H$ for some Henkin set H. Moreover, $w \xrightarrow{\tau} w'$ because $B^\tau \in H$ whenever for $\Box B \in w$.

Now for logics without substitution.

Let $S_1, S_2 \ldots$ enumerate all sentences in w of the form

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B,$$

where $x_1, \ldots, x_n$ are zero or more distinct variables free in B. Let U be the "unused" $\mathcal{L}$-variables that are not in the range of τ. Let Z be an infinite subset of U such that $Z \backslash U$ is also infinite. For each $S_i = (x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B)$, let Z_{S_i} be a set of distinct variables $z_1, \ldots, z_n \in Z$ such that $Z_{S_i} \cap \bigcup_{j<i} Z_{S_j} = \emptyset$ (i.e. none of the z_i has been used for any earlier

S_j). Abbreviate

$$\begin{aligned}
B_i &=_{df} [z_1, \ldots, z_n/x_1^\tau, \ldots, x_n^\tau]B^\tau;\\
X_i &=_{df} x_1 = y_1 \wedge \ldots \wedge x_n = y_n;\\
Y_i &=_{df} y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau;\\
Z_i &=_{df} y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n.
\end{aligned}$$

(For $n = 0$, X_i, Y_i and Z_i are the tautology $\top$, and B_i is B^τ.)

Let $\Gamma^- = \{(Y_i \supset Z_i \wedge B_i) : S_i \in S_1, S_2, \ldots\}$, and let $\Gamma = \Gamma^- \cup \{A^\tau\}$.

Suppose for reductio that Γ is inconsistent. Then there are sentences $(Y_1 \supset Z_1 \wedge B_1), \ldots, (Y_m \supset Z_m \wedge B_m) \in \Gamma^-$ such that

(1) $\quad \vdash_L \neg(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m))$.

By (Nec),

(2) $\quad \vdash_L \Box\neg(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m))$.

Any member $(Y_i \supset Z_i \wedge B_i)$ of Γ^- has the form

$$y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n/x_1^\tau, \ldots, x_n^\tau]B^\tau.$$

By ($\mathrm{CS_n}$),

(3) $\quad \vdash_L x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau \wedge \Box B^\tau \supset$
$\Box(y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \supset [z_1, \ldots, z_n/x_1^\tau, \ldots, x_n^\tau]B^\tau)$.

Now w contains $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B$. So w^τ contains $x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau \wedge \Box B^\tau$, which is the antecedent of (3). The consequent of (3) is $\Box(Z_i \supset B_i)$. Thus

(4) $\quad w^\tau \vdash_L \Box(Z_1 \supset B_1) \wedge \ldots \wedge \Box(Z_m \supset B_m)$.

Let $\Delta = w^\tau \cup \{\Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))\}$. So

(5) $\quad \Delta \vdash_L \Box(Z_1 \supset B_1) \wedge \ldots \wedge \Box(Z_m \supset B_m)$;

(6) $\quad \Delta \vdash_L \Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))$.

By (K) and (Nec), (5) and (6) yield

(7) $\quad \Delta \vdash_L \Diamond(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m))$.

By (2), it follows that Δ is inconsistent. This means that

(8) $\quad w^\tau \vdash_L \neg\Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))$.

Now consider $Z_1 = (y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n)$. By ($\mathrm{LL_n^*}$) (or repeated application of ($\mathrm{LL^*}$)),

(9) $\quad \vdash_L y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \supset \Box\neg(A^\tau \wedge (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n))$
$\supset \Box\neg(A^\tau \wedge (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau))$,

because the z_i are not free in A^τ. In other words (and dropping the tautologous conjunct at the end),

(10) $\vdash_L Z_1 \supset \Box\neg(A^\tau \wedge (Y_1 \supset Z_1)) \supset \Box\neg A^\tau$.

By the same reasoning,

(11) $\vdash_L Z_1 \wedge \ldots \wedge Z_m \supset \Box\neg(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m)) \supset \Box\neg A^\tau$.

By (PC), (Nec) and (K), this means

(12) $\vdash_L Z_1 \wedge \ldots \wedge Z_m \supset \Diamond A^\tau \supset \Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))$.

Since $w^\tau \vdash_L \Diamond A^\tau$, (8) and (12) together entail

(13) $w^\tau \vdash_L \neg(Z_1 \wedge \ldots \wedge Z_m)$.

So there are $C_1, \ldots, C_k \in w$ such that

(14) $\vdash_L C_1^\tau \wedge \ldots \wedge C_k^\tau \supset \neg(Z_1 \wedge \ldots \wedge Z_m)$.

Each Z_i has the form $y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n$. All the z_i are pairwise distinct, and none of them occur in $C_1^\tau \wedge \ldots \wedge C_k^\tau$ (because the z_i are not in the range of τ) nor in any other Z_i. By (Sub*), we can therefore replace each z_i in (14) by the corresponding y_i^τ, turning Z_i into Y_i:

(15) $\vdash_L C_1^\tau \wedge \ldots \wedge C_k^\tau \supset \neg(Y_1 \wedge \ldots \wedge Y_m)$.

For any $Y_i = (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau)$, X_i is a sentence of the form $x_1 = y_1 \wedge \ldots \wedge x_n = y_n$. So X_i^τ is $x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau$, and $\vdash_L X_i^\tau \supset Y_i$ by either (=R) or (Neg) and ($\forall$=R). So (15) entails

(16) $\vdash_L C_1^\tau \wedge \ldots \wedge C_k^\tau \supset \neg(X_1^\tau \wedge \ldots \wedge X_m^\tau)$.

Thus by (Sub$^\tau$),

(17) $\vdash_L C_1 \wedge \ldots \wedge C_k \supset \neg(X_1 \wedge \ldots \wedge X_m)$.

Since $\{C_1, \ldots, C_k, X_1, \ldots, X_m\} \subseteq w$, it follows that w is inconsistent. Which it isn't. This completes the reductio.

So Γ is consistent. Since the infinitely many variables in $U \backslash Z$ do not occur in Γ, lemma 7.7 guarantees that $\Gamma \subseteq w'$ for some world w' in the canonical model for L. And of course, Γ was constructed so that w' satisfies the condition in definition 7.4 for $w \xrightarrow{\tau} w'$. This requires that for every formula B and variables $x_1 \ldots x_n$, $y_1, \ldots, y_n$ such that the $x_1 \ldots x_n$ are zero or more pairwise distinct members of $\mathit{Varf}(B)$, if $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B \in w$ and $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \in w'$, then there are variables $z_1 \ldots z_n \notin \mathit{Var}(B^\tau)$ such that $z_1 = y_1^\tau \wedge \ldots \wedge z_n = y_n^\tau \wedge [z_1 \ldots z_n / x_1^\tau \ldots x_n^\tau]B^\tau \in w'$. By construction of Γ, whenever $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B \in w$, then there are suitable $z_1, \ldots, z_n$ such that $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]B^\tau \in w'$. So if $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \in w'$, then $y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]B^\tau \in w'$. ■

Lemma 7.9 (Truth Lemma)
For any sentence A and world w in the canonical model $\mathcal{M}_L = \langle W, R, U, D, K, V\rangle$ for L,

$$w, V \Vdash A \text{ iff } A \in w.$$

Proof by induction on A.

1. A is $Px_1 \dots x_n$. $w, V \Vdash Px_1 \dots x_n$ iff $\langle V_w(x_1), \dots, V_w(x_n)\rangle \in V_w(P)$ by definition 2.7. By construction of V_w (definition 7.5), $V_w(x_i)$ is $[x_i]_w$ or undefined if $[x_i]_w = \emptyset$, and $V_w(P) = \{\langle [z_1]_w, \dots, [z_n]_w\rangle : Pz_1 \dots z_n \in w\}$. (For non-logical P, this is directly given by definition 7.5; for the identity predicate, $V_w(=)$ is $\{\langle d, d\rangle : d \in U_w\}$ by definition 2.7, which equals $\{\langle [z]_w, [z]_w\rangle : z{=}z \in w\} = \{\langle [z_1]_w, [z_2]_w\rangle : z_1{=}z_2 \in w\}$ because the members of U_w are precisely the non-empty sets $[z]_w$.)

   Now if $\langle V_w(x_1), \dots, V_w(x_n)\rangle \in V_w(P)$, then $\langle [x_1]_w, \dots, [x_n]_w\rangle \in \{\langle [z_1]_w, \dots, [z_n]_w\rangle : Pz_1 \dots z_n \in w\}$, where all the $[x_i]_w$ are non-empty (for $V_w(x_i)$ is defined). This means that there are variables $z_1, \dots, z_n$ such that $\{x_1{=}z_1, \dots, x_n{=}z_n, Pz_1 \dots z_n\} \subseteq w$. Then $Px_1 \dots x_n \in w$ by (LL*).

   In the other direction, if $Px_1 \dots x_n \in w$, then $x_i = x_i \in w$ for all x_i in $x_1 \dots x_n$ (see p. 78). Hence $\langle [x_1]_w, \dots, [x_n]_w\rangle \in \{\langle [z_1]_w, \dots, [z_n]_w\rangle : Pz_1 \dots z_n \in w\}$, i.e. $\langle V_w(x_1), \dots, V_w(x_n)\rangle \in V_w(P)$.

2. A is $\neg B$. $w, V \Vdash \neg B$ iff $w, V \nVdash B$ by definition 2.7, iff $B \notin w$ by induction hypothesis, iff $\neg B \in w$ by maximality of w.

3. A is $B \supset C$. $w, V \Vdash B \supset C$ iff $w, V \nVdash B$ or $w, V \Vdash C$ by definition 2.7, iff $B \notin w$ or $C \in w$ by induction hypothesis, iff $B \supset C \in w$ by maximality and consistency of w and the fact that $\vdash_L \neg B \supset (B \supset C)$ and $\vdash_L C \supset (B \supset C)$.

4. A is $\langle y : x\rangle B$. Assume first that $w, V \Vdash y{\neq}y$. So $V_w(y)$ is undefined, and it is not the case that $V_w(y)$ has multiple counterparts at any world. And then $w, V \Vdash \langle y : x\rangle B$ iff $w, V^{[y/x]} \Vdash B$ by definition 3.2, iff $w, V \Vdash [y/x]B$ by lemma 3.9, iff $[y/x]B \in w$ by induction hypothesis. Also by induction hypothesis, $y{\neq}y \in w$. By (SCN), $\vdash_L y{\neq}y \supset ([y/x]B \leftrightarrow \langle y : x\rangle B)$. So $[y/x]B \in w$ iff $\langle y : x\rangle B \in w$.

   Next, assume that $w, V \Vdash y{=}y$; so by induction hypothesis $y{=}y \in w$. Assume further that $\langle y : x\rangle B \notin w$. Then $\neg\langle y : x\rangle B \in w$ by maximality of w, and $\langle y : x\rangle \neg B \in w$ by (S¬). Since w is substitutionally witnessed and $y = y \in w$, there is a variable $z \notin \mathit{Var}(\langle y : x\rangle \neg B)$ such that $y{=}z \in w$ and $[z/x]\neg B \in w$. By induction hypothesis, $w, V \Vdash y = z$. Moreover, by definition 3.3, $\neg[z/x]B \in w$, and so $[z/x]B \notin w$ by consistency of w. By induction hypothesis, $w, V \nVdash [z/x]B$. By definition 2.7, then $w, V \Vdash \neg[z/x]B$, i.e. $w, V \Vdash [z/x]\neg B$. Since z and x are modally separated in B, then $w, V^{[z/x]} \Vdash \neg B$ by lemma 3.9. But $V^{[z/x]}$ and $V^{[y/x]}$ agree on all variables at w, because $w, V \Vdash y{=}z$. So $w, V^{[y/x]} \Vdash \neg B$ by the locality lemma 2.10. So $w, V^{[y/x]} \nVdash B$ by definition 2.7, and $w, V \nVdash \langle y : x\rangle B$ by definition 3.2.

In the other direction, assume $\langle y : x\rangle B \in w$. Since w is substitutionally witnessed and $y{=}y \in w$, there is a new variable z such that $y{=}z \in w$ and $[z/x]B \in w$. By induction hypothesis, $w, V \Vdash y{=}z$ and $w, V \Vdash [z/x]B$. Since z and x are modally separated in B, $w, V^{[z/x]} \Vdash B$ by lemma 3.9. As before $V^{[z/x]}$ and $V^{[y/x]}$ agree on all variables at w, because $w, V \Vdash y{=}z$; so $w, V^{[y/x]} \Vdash B$ by lemma 2.10 and $w, V \Vdash \langle y : x\rangle B$ by definition 3.2.

5. A is $\forall xB$. We first show that for any variable x, $w, V \Vdash Ex$ iff $Ex \in w$: $w, V \Vdash Ex$ iff $V_w(x) \in D_w$ by definition 3.3, iff $[x]_w \in D_w$ by definition 7.5, iff $Ex \in w$ by definition 7.5.

   Now assume $\forall xB \in w$, and let y be any variable such that $Ey \in w$. As just shown, $w, V \Vdash Ey$. By (FUI**), $\exists x(x = y \land B) \in w$. By witnessing, there is a $z \notin \mathit{Var}(B)$ such that $z = y \land [z/x]B \in w$, and thus $z = y \in w$ and $[z/x]B \in w$. By induction hypothesis, $w, V \Vdash z{=}y$ and $w, V \Vdash [z/x]B$. By lemma 3.9, then $w, V^{[z/x]} \Vdash B$. And since $V_w(z) = V_w(y)$, it follows by lemma 2.10 that $w, V^{[y/x]} \Vdash B$. So if $\forall xB \in w$, then $w, V^{[y/x]} \Vdash B$ for all variables y with $Ey \in w$, i.e. with $V_w(y) \in D_w$. Since every member $[y]_w$ of D_w is denoted by some variable y under V_w, this means that $w, V' \Vdash B$ for all existential x-variants V' of V on w. So $w, V \Vdash \forall xB$.

   Conversely, assume $\forall xB \notin w$. Then $\exists x\neg B \in w$; so by witnessing, $[y/x]\neg B \in w$ for some $y \notin \mathit{Var}(B)$ with $Ey \in w$. Then $\neg[y/x]B \in w$ and so $[y/x]B \notin w$. As shown above, $w, V \Vdash Ey$. Moreover, by induction hypothesis, $w, V \nVdash [y/x]B$. By lemma 3.9, then $w, V^{[y/x]} \nVdash B$. Let V' be the (existential) x-variant of V on w with $V'_w(x) = V^{[y/x]}_w(x)$. By the locality lemma, $w, V' \nVdash B$. So $w, V \nVdash \forall xB$.

6. A is $\Box B$. Assume $w, V \Vdash \Box B$. Then $w', V' \Vdash B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. We first show that if $w \xrightarrow{\tau} w'$, then $V_w \triangleright V^\tau_{w'}$. By definitions 2.6 and 7.5, $V_w \triangleright V^\tau_{w'}$ means that there is a transformation σ such that $w \xrightarrow{\sigma} w'$ and for every variable y, if there is a $z \in V_w(y)$ such that $[z^\sigma]_{w'} \in U_{w'}$ (i.e., if $V_w(y)$ has any σ-counterpart at w'), then there is a $z \in V_w(y)$ with $z^\sigma \in V^\tau_{w'}(y)$ (i.e., then $V^\tau_{w'}(y)$ is such a counterpart), otherwise $V^\tau_{w'}(y)$ is undefined. The relevant transformation σ will be τ. So what we'll show is this: for every variable y, if there is a $z \in V_w(y)$ such that $[z^\tau]_{w'} \in U_{w'}$, then there is a $z \in V_w(y)$ with $z^\tau \in V^\tau_{w'}(y)$, otherwise $V^\tau_{w'}(y)$ is undefined.

   Let y be any variable. Assume first that there is a $z \in V_w(y)$ such that $[z^\tau]_{w'} \in U_{w'}$. Then $z{=}y \in w$ and $z^\tau{=}z^\tau \in w'$. By either (Neg) and (EI) or (=R), $\vdash_L z{=}y \supset y{=}y$; so $y = y \in w$. Moreover, by either (TE), (EI), (Nec) and (K) or (=R) and (Nec), $\vdash_L z = y \supset \Box(z = z \supset y = y)$; so $\Box(z = z \supset y = y) \in w$. By definition of $w \xrightarrow{\tau} w'$, then $z^\tau = z^\tau \supset y^\tau = y^\tau \in w'$. So $y^\tau = y^\tau \in w'$. Hence $y \in V_w(y)$ and $y^\tau \in [y^\tau]_{w'} = V_{w'}(y^\tau) = V^\tau_{w'}(y)$. Alternatively, assume there is no $z \in V_w(y)$ with $z^\tau = z^\tau \in w'$. Then either $V_w(y) = \emptyset$, in which case $y \neq y \in w$, and so $\Box(y \neq y) \in w$ by (NA), (EI), (Nec) and (K), and $y^\tau \neq y^\tau \in w'$ by definition of $w \xrightarrow{\tau} w'$, or else $V_w(y) \neq \emptyset$, but $z^\tau \neq z^\tau \in w'$ for all $z \in V_w(y)$, in which case, too, $y^\tau \neq y^\tau \in w'$ since $y \in V_w(y)$. Either way, $V_{w'}(y^\tau) = V^\tau_{w'}(y)$ is undefined.

   We've shown that if $w, V \Vdash \Box B$, then for every w' and τ with $w \xrightarrow{\tau} w'$, $w', V^\tau \Vdash B$. By the transformation lemma, then $w', V \Vdash B^\tau$. By induction hypothesis, $B^\tau \in w'$. Now suppose $\Box B \notin w$. Then $\Diamond\neg B \in w$ by maximality of w. By the existence lemma, there is then a world w' and transformation τ with $w \xrightarrow{\tau} w'$ and $\neg B^\tau \in w'$. (Any transformation whose range excludes infinitely many variables will do.) But we've just seen that if $w \xrightarrow{\tau} w'$, then $B^\tau \in w'$. So if $w, V \Vdash \Box B$, then $\Box B \in w$.

For the other direction, assume $w, V \nVdash \Box B$. So $w', V' \nVdash B$ for some w', V' with wRw' and $V_w \triangleright V'_{w'}$. As before, $V_w \triangleright V'_{w'}$ means that there is a transformation τ with $w \xrightarrow{\tau} w'$ such that for every variable x, either there is a $y \in V_w(x)$ with $y^\tau \in V'_{w'}(x)$, or there is no $y \in V_w(x)$ with $y^\tau = y^\tau \in w'$, in which case $V'_{w'}(x)$ is undefined. Let τ be any transformation with $w \xrightarrow{\tau} w'$, and let $*$ be a substitution that maps each variable x in B to some member y of $V_w(x)$ with $y^\tau \in V'_{w'}(x)$, or to itself if there is no such y. Thus if $x \in \mathit{Var}(B)$ and $V'_{w'}(x)$ is defined, then $(*x)^\tau \in V'_{w'}(x)$, and so $V'_{w'}(x) = [(*x)^\tau]_{w'} = V^{\tau;*}_{w'}(x)$. Alternatively, if $V'_{w'}(x)$ is undefined (so $*x = x$), then $V^{\tau;*}_{w'}(x) = V^\tau_{w'}(x)$ is also undefined. The reason is that otherwise $V^\tau_{w'}(x) = [x^\tau]_{w'} \neq \emptyset$ and $x^\tau = x^\tau \in w'$; by definition of accessibility, then $\Box x \neq x \notin w$ and hence $x = x \in w$, as $\vdash_L x \neq x \supset \Box x \neq x$; so there is a $y \in V_w(x)$, namely x, such that $y^\tau = y^\tau \in w'$, in which case $V'_{w'}(x)$ cannot be undefined (by definition 2.6). So V' and $V^{\tau\cdot *}$ agree at w' on all variables in B. By lemma 2.9, $w', V^{\tau\cdot *} \nVdash B$.

Now suppose for reductio that $\Box B \in w$. Let $x_1, \ldots, x_n$ be the variables x in $\mathit{Var}(B)$ with $(*x)^\tau \in V'_{w'}(x)$ (thus excluding empty variables as well as variables denoting individuals without τ-counterparts at w'). For each such x_i, $*x_i \in V_w(x_i)$, and so $x_i = *x_i \in w$. If L is with substitution, then by ($\mathrm{LL_n}$), $\langle *x_1, \ldots, *x_n : x_1, \ldots, x_n \rangle \Box B \in w$; so $\Box \langle *x_1, \ldots, *x_n : x_1, \ldots, x_n \rangle B \in w$ by (S$\Box$). By definition of $w \xrightarrow{\tau} w'$, then $\langle (*x_1)^\tau, \ldots, (*x_n)^\tau : x_1^\tau, \ldots, x_n^\tau \rangle B^\tau \in w'$. By substitutional witnessing, it follows that there are new variables $z_1, \ldots, z_n$ such that $z_i = (*x_i)^\tau \in w'$ and (hence) $[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$. If L is without substitution, this fact – that there are new variables $z_1, \ldots, z_n$ such that $z_i = (*x_i)^\tau \in w'$ and $[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$ – is guaranteed directly by definition of $w \xrightarrow{\tau} w'$ and the fact that $\Box B \in w$.

By induction hypothesis, $w', V \Vdash z_i = (*x_i)^\tau$ and $w', V \Vdash [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau$. Since the z_i are new, $w', V^{[z_1,\ldots,z_n/x_1^\tau,\ldots,x_n^\tau]} \Vdash B^\tau$ by lemma 3.9. By the transformation lemma 3.13, then $w', V^{[z_1,\ldots,z_n/x_1^\tau,\ldots,x_n^\tau]\cdot\tau} \Vdash B$. However, for each x_i, $V^{[z_1,\ldots,z_n/x_1^\tau,\ldots,x_n^\tau]\cdot\tau}_{w'}(x_i) = V^{[z_1,\ldots,z_n/x_1^\tau,\ldots,x_n^\tau]}_{w'}(x_i^\tau) = V_{w'}(z_i) = V_{w'}((*x_i)^\tau)$ (because $w', V \Vdash (*x_i)^\tau = z_i$) $= V^\tau_{w'}(*x_i) = V^{\tau\cdot[*x_1,\ldots,*x_n/x_1,\ldots,x_n]}_{w'}(x_i) = V^{\tau;*}_{w'}(x_i)$. Similarly, if $x \in \mathit{Var}(B)$ is none of the $x_1, \ldots, x_n$, so $(*x)^\tau \notin V'_{w'}(x)$, then $*x$ is x by definition of $*$, and so $V^{[z_1,\ldots,z_n/x_1^\tau,\ldots,x_n^\tau]\cdot\tau}_{w'}(x) = V^\tau_{w'}(x) = V^\tau_{w'}(*x) = V^{\tau;*}_{w'}(x)$. So $V^{[z_1,\ldots,z_n/x_1^\tau,\ldots,x_n^\tau]\cdot\tau}$ and $V^{\tau;*}_{w'}$ agree at w' on all variables in B. By lemma 2.10, then $w', V^{\tau\cdot *} \Vdash B$ – contradiction. ■

8 Completeness

Recall that a logic L in some language of quantified modal logic is *(strongly) complete with respect to a class of models* $\mathbb{M}$ if every L-consistent set of formulas Γ is verified at some world in some model in $\mathbb{M}$. L is *characterised by* $\mathbb{M}$ if L is sound and complete with respect to $\mathbb{M}$.

The minimal positive and negative logics from sections 4 and 5 were designed to be complete with respect to the class of all positive and negative models, respectively. Let's confirm that this is the case.

THEOREM 8.1 (COMPLETENESS OF P AND P_s)
The logics P and P_s are (strongly) complete with respect to the class of positive counterpart models.

PROOF Let L range over P and P_s. We have to show that whenever a set of $\mathcal{L}$-formulas Γ is L-consistent, then there is some world in some positive counterpart model that verifies all members of Γ. By lemma 7.6, the canonical model $\mathcal{M}_L = \langle \mathcal{S}_L, V_L \rangle$ for L is a positive model. By the Extensibility Lemma, $\Gamma \subseteq w$ for some world w in $\mathcal{M}_L$, since none of the infinitely many variables Var^+ occur in Γ. By the truth lemma, then $w, V_L \Vdash_{\mathcal{S}_L} A$ for each $A \in \Gamma$. ∎

THEOREM 8.2 (COMPLETENESS OF N AND N_s)
The logics N and N_s are (strongly) complete with respect to the class of negative counterpart models.

PROOF Let L range over N and N_s, and let Γ be an L-consistent set of $\mathcal{L}$-formulas. By lemma 7.6, the canonical model $\mathcal{M}_L = \langle \mathcal{S}_L, V_L \rangle$ for L is a negative model. By the Extensibility Lemma, $\Gamma \subseteq w$ for some world w in $\mathcal{M}_L$, since none of the infinitely many variables Var^+ occur in Γ. By the truth lemma, then $w, V_L \Vdash_{\mathcal{S}_L} A$ for each $A \in \Gamma$. ∎

Together with the soundness theorems 4.3, 4.6, 5.4 and 5.5, it follows that P and P_s are characterized by the class of positive models, and N and N_s by the class of negative models.

In footnote 1 (on page 5) I mentioned that the introduction of multiple counterpart relations makes little difference to the base logic. Let's call a counterpart structure in which any two worlds are linked by at most one counterpart relation *unirelational*. As it turns out, P and P_s are also characterized by the class of unirelational positive models, and N and N_s by the class of unirelational negative models. The easiest way to see this is perhaps to note that all the lemmas in the previous section still go through if we define accessibility and counterparthood in canonical models by a fixed transformation τ whose range excludes infinitely many variables. The extensibility lemma 7.7 and existence lemma 7.8 are unaffected by this change; the only part that needs adjusting is the clause for $\Box B$ in the proof of the truth lemma 7.9, but the adjustments are straightforward.

Every quantified modal logic is strongly complete with respect to every class of models that contains its canonical model. However, on the traditional idea that logical truths should be true on any interpretation of the non-logical terms, an arguably more important kind of completeness is completeness with respect to all models with a certain type of *structure*. Strictly speaking, we have two such notions, one for positive and one for negative logics. As in section **??**, let me focus on positive logics for the moment.

> Definition 8.3 (Positive completeness and characterisation)
> A logic L in some language of quantified modal logic is *(strongly) positively complete with respect to a class of structures* $\mathbb{S}$ if every L-consistent set of formulas Γ is verified at some world in some positive model $\langle \mathcal{S}, V \rangle$ with $\mathcal{S} \in \mathbb{S}$. L is *positively characterised by* $\mathbb{S}$ if it is sound and positively complete with respect to $\mathbb{S}$.

Now we might try to show, first, that if a PML is categorical, then so is its quantified counterpart (whatever that means). Then we could show that the canonical model of the PML is in a class of frames F iff the opaque propositional guise of the canonical model of the quantified counterpart is in F. Then we'd have completeness transfer for all categorical logics.

> Theorem 8.4 ((Positive) completeness transfer)
> If L is a (unimodal) propositional modal logic that is complete with respect to the Kripke frames in some class F, then the PL is positively complete with respect to the total counterpart structures whose opaque propositional guise is in F.

One way to
[To be continued...]

9 Locally classical and two-dimensional logics

10 Individualistic semantics

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